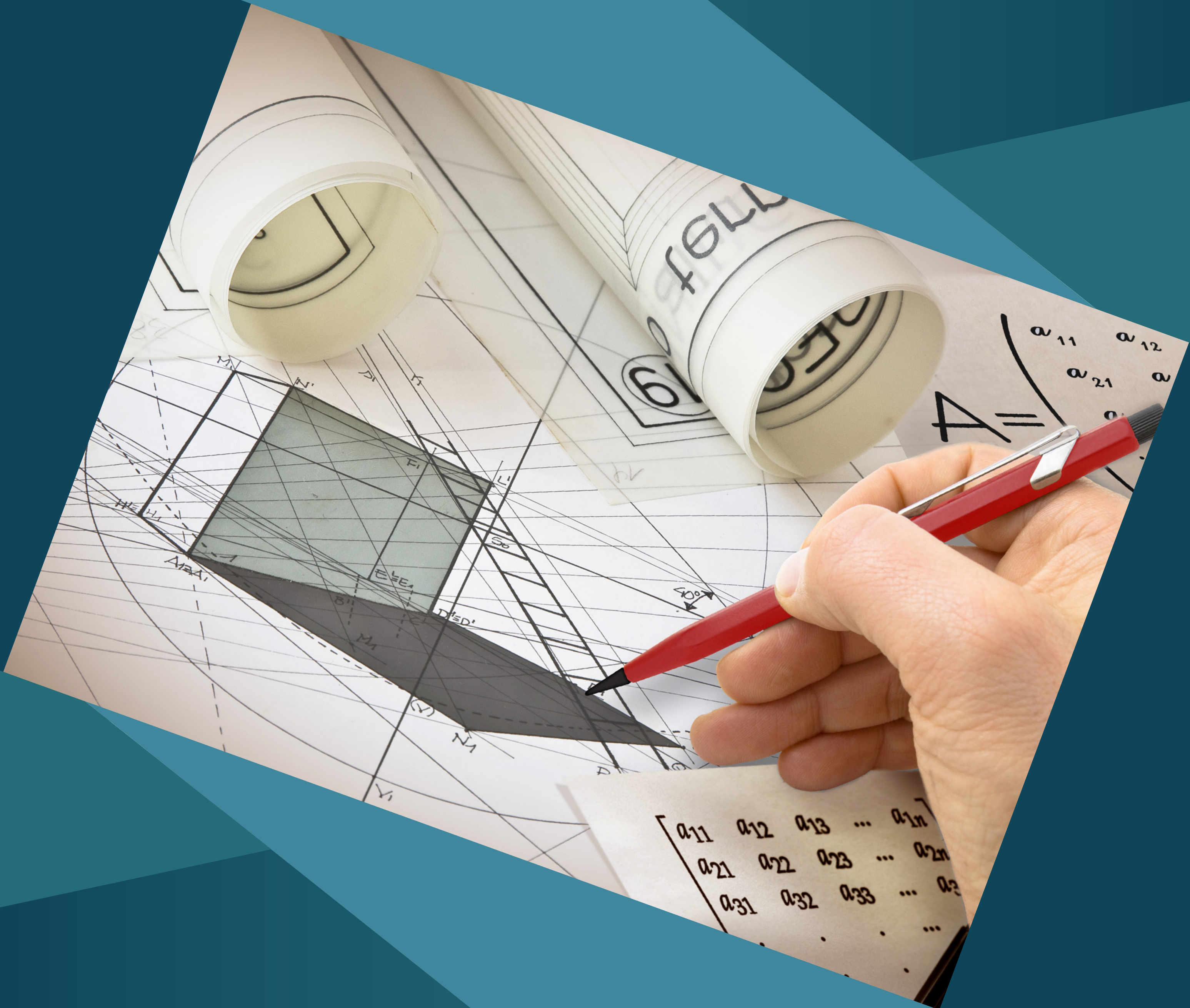

# PRACTICAL GUIDEBOOK FOR MATHEMATICS TEACHING IN HIGHER EDUCATION

Linear and Vector Algebra

# Practical Guidebook for Mathematics Teaching in Higher Education

## Linear and Vector Algebra

RTU Press

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# CONTENT

Elements of Linear Algebra

Introduction

Linear algebra is a branch of mathematics that deals with vectors, matrices, and systems of linear equations. It focuses on understanding and manipulating linear relationships – those in which variables change proportionally. At the heart of linear algebra are matrix operations, which allow us to model and solve problems involving multiple variables efficiently. These concepts are essential not only in pure mathematics but also in various applied fields. In engineering, linear algebra is used to analyze electrical circuits and control systems; in computer science, it forms the foundation of computer graphics, machine learning, and cryptography; in physics, it helps describe physical systems and transformations; and in economics, it is used in optimization and modeling input-output relationships. Due to its wide applicability and logical structure, linear algebra is a fundamental subject for students across many disciplines.

Beginning over 4000 years ago, the Babylonians began exploring ways to apply mathematics to everyday tasks, which contributed to their development as a leading civilization. Since the early 19th century, archaeologists have uncovered around 500 000 Babylonian clay tablets, though fewer than 500 of them contain mathematical content. By the late 1800s, scholars had translated these mathematical texts, providing us with valuable insight into the Babylonians' mathematical knowledge and how they employed fundamental concepts in practice.

The Nine Chapters on the Mathematical Art is an ancient Chinese mathematical text developed over several centuries, from the 10th to the 2nd century BCE, with its final version dating to the 1st century CE. It is among the oldest surviving mathematical works from China, alongside the Suan shu shu (202–186 BCE) and the Zhoubi Suanjing (compiled during the Han dynasty and into the late 2nd century CE). This text presents a practical, problem-solving approach to mathematics, emphasizing general methods for solving various problems. This contrasts with the style of ancient Greek mathematicians, who typically derived results from a foundational set of axioms.

The concept of determinants was developed independently in two parts of the world: in 1683 by Seki Takakazu in Japan, and around the same time, in 1693, by Gottfried Wilhelm Leibniz in Europe. Later, in 1750, Gabriel Cramer introduced what is now known as Cramer's Rule, although he did not provide a formal proof. Both Cramer and Étienne Bézout, who contributed in 1779, were drawn to the study of determinants while investigating the problem of finding plane curves that pass through a given set of points.

In 1771, Vandermonde was the first to recognize determinants as independent mathematical functions, rather than just tools for solving systems of equations. Shortly after, in 1772, Pierre-

Simon Laplace introduced a general method for expanding a determinant using its complementary minors – a technique that became foundational in determinant theory.

The next significant advancement came from Carl Friedrich Gauss in 1801. Like Joseph-Louis Lagrange, Gauss extensively used determinants in number theory. He introduced the term "determinant", although at that time he used it to refer specifically to the discriminant of a quadratic form, not in the modern sense of a general matrix function.

In 1812, Cauchy made a major contribution to the theory of determinants. He used the word "determinant" in its present sense –summarized and simplified what was then known on the subject, improved the notation, and gave the multiplication theorem with a proof. With him begins the theory in its generality.

In 1841, Jacobi used the functional determinant, which Sylvester later called the Jacobian.

By the 20th century, linear algebra had become a central field in mathematics, closely linked to abstract algebra and vector spaces. Today, it is not only a foundational mathematical discipline but also a crucial tool in science, engineering, and data analysis.

Famous scientists in the field of Linear algebra

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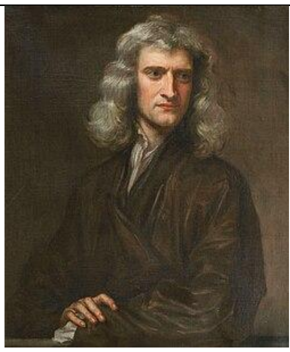
|  | **Isaac Newton (1643–1727)** Though best known for calculus and physics, he worked with systems of equations and determinants long before formal matrix theory. |  | **Leonhard Euler (1707–1783)** contributed to determinants and systems of linear equations; laid early groundwork for matrix methods. |
|  | **Carl Friedrich Gauss (1777–1855)** developed Gaussian elimination for solving systems of linear equations. |  | **Augustin-Louis Cauchy (1789–1857)** formalized the concept of determinants and introduced the characteristic equation of a matrix. |

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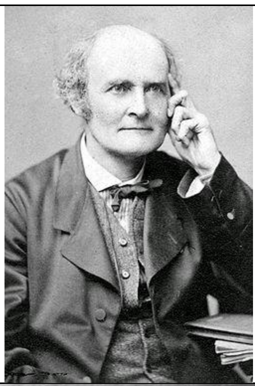
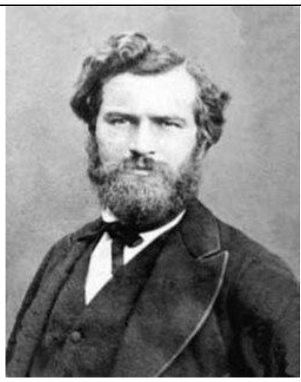
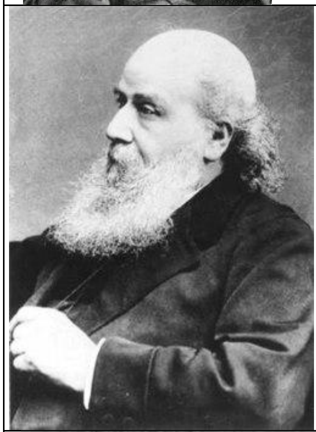
|  **Arthur Cayley (1821–1895)**<br>the founder of matrix theory, introduced matrix multiplication and matrix groups. |  **Camille Jordan (1838–1922)**<br>known for the Jordan canonical form, a key concept in linear algebra and differential equations. |
|  **James Joseph Sylvester (1814–1897)**<br>coined the term “matrix”, contributed to invariant theory and linear transformations. |  **Évariste Galois (1811–1832)**<br>laid foundations for linking group theory and linear algebra**.** |
|  **Hermann Grassmann (1809–1877)**<br>introduced the concepts of vector spaces and exterior algebra. |  **David Hilbert (1862–1943)**<br>developed Hilbert spaces, central in linear algebra and functional analysis. |
|  **Emmy Noether (1882–1935)**<br>made groundbreaking contributions to abstract algebra and invariant theory. |  **John von Neumann (1903–1957)**<br>applied linear algebra to quantum mechanics; developed von Neumann algebras. |

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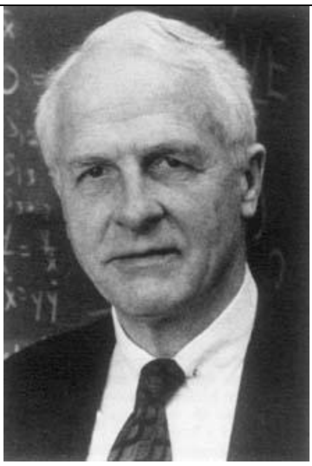
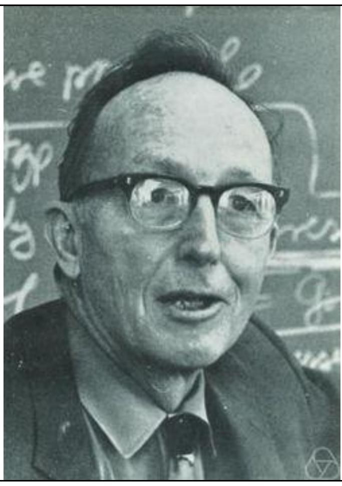
|  | **Garrett Birkhoff (1911–1996)** co-authored the influential textbook *Linear Algebra*. |  | **Saunders Mac Lane (1909–2005)** helped formalize modern algebra; co-founder of category theory**.** |

# Chapter 1: Linear Algebra

## 1.1. The Concept of a Matrix

### Definition of a matrix

A matrix is an $m \times n$ ($m$ by $n$) array or table of numbers or other mathematical objects with elements or entries arranged in rows and columns:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

*Examples:*

$$\begin{pmatrix} 1 & -3 \\ 4 & 5 \end{pmatrix}; \begin{pmatrix} 9 & 0 \\ -2 & 7 \\ 1 & 3 \end{pmatrix}; \begin{pmatrix} \frac{1}{2}; & 0; & -1 \end{pmatrix}; \begin{pmatrix} \cos x & \sin x & -\sin x \\ \operatorname{tg} x & \operatorname{ctg} x & \operatorname{tg} x \\ \sin x & -\cos x & 1 \end{pmatrix}.$$

The entries in a matrix are called the components of the matrix and can be written as $a_{ij}$, where $i$ indicates the row number and runs from 1 to $m$, and $j$ indicates the column number and runs from 1 to $n$.
We generally use uppercase letters from the beginning of the alphabet (A, B, C…) to denote matrices.
A matrix is identified with its components. Given matrix A with components $a_{ij}$, $1 \leq i \leq m$, $1 \leq j \leq n$, we may write $A = [a_{ij}]$.
The symbol $a_{12}$ represents the element in the first row of the second column.
The main diagonal of a matrix is the diagonal consisting of the elements $a_{11}, a_{22}, \dots, a_{mn}$.
A vector is a $1 \times n$ or $n \times 1$ matrix. That is an ordered set of $n$ numbers. We say that such a vector is of dimension $n$.
We generally use lowercase letters (a, b, c…) to denote vectors.

## 1.2. Types of matrices

### Square matrix

A square matrix has the same number of rows as columns:

$$A_{n\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

### Diagonal matrix

A diagonal matrix is a square matrix with zeros everywhere except on the diagonal, which runs from top left to bottom right:

$$D = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}.$$

**Zero matrix**

The zero matrix is an $m \times n$ matrix, all of whose entries are 0:

$$Z = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

**Identity matrix**

An identity matrix is a diagonal matrix with all its diagonal elements equal to 1 and zeros everywhere else. The letter $I$ is usually used to label identity matrices:

$$I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

**Upper triangular matrix**

A square matrix is called upper triangular if all the elements below the main diagonal are equal to zero:

$$U = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}.$$

**Lower triangular matrix**

A square matrix is called lower triangular if all the elements above the main diagonal are equal to zero:

$$L = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}.$$

A matrix with only one column is called a column matrix, and a matrix with only one row is called a row matrix:

$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}$ – a column matrix; $B = (b_1 \quad b_2 \ \dots \quad b_n)$ – a row matrix.

## 1.3. Operations with matrices

### 1.3.1. Matrix addition and subtraction

Two matrices must be compatible for them to be added or subtracted. Matrices are said to be compatible when they have the same size.

The sum (or difference) of two compatible matrices is a matrix of the same size, with each element the sum (or difference) of the corresponding elements of the two matrices:

$$A \pm B = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \pm \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix} =$$

$$= \begin{pmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & \dots & a_{1n} \pm b_{1n} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & \dots & a_{2n} \pm b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} \pm b_{m1} & a_{m2} \pm b_{m2} & \dots & a_{mn} \pm b_{mn} \end{pmatrix}.$$

*Properties*

Let A, B, and C be matrices of the same size

Matrix addition is commutative:

$A + B = B + A.$

Matrix addition is associative:

$$(A + B) + C = A + (B + C);$$

$$A + 0 = A.$$

*Example*

Let $A = \begin{pmatrix} 4 & 0 & -3 \\ 1 & -7 & 2 \\ 0 & -9 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -2 & 0 \\ -5 & 4 & 5 \\ 0 & -6 & 3 \end{pmatrix}$, then calculate

a) $A + B$
b) $B - A$

*Solution*

a) $A+B=\begin{pmatrix}4 & 0 & -3\\ 1 & -7 & 2\\ 0 & -9 & -1\end{pmatrix}+\begin{pmatrix}1 & -2 & 0\\ -5 & 4 & 5\\ 0 & -6 & 3\end{pmatrix}=\begin{pmatrix}4+1 & 0+(-2) & -3+0\\ 1+(-5) & -7+4 & 2+5\\ 0+0 & -9+(-6) & -1+3\end{pmatrix}=$
$=\begin{pmatrix}5 & -2 & -3\\ -4 & -3 & 7\\ 0 & -15 & 2\end{pmatrix};$

b) $B-A=\begin{pmatrix}1 & -2 & 0\\ -5 & 4 & 5\\ 0 & -6 & 3\end{pmatrix}-\begin{pmatrix}4 & 0 & -3\\ 1 & -7 & 2\\ 0 & -9 & -1\end{pmatrix}=\begin{pmatrix}1-4 & -2-0 & 0-(-3)\\ -5-1 & 4-(-7) & 5-2\\ 0-0 & -6-(-9) & 3-(-1)\end{pmatrix}=$
$=\begin{pmatrix}-3 & -2 & 3\\ -6 & 11 & 3\\ 0 & 3 & 4\end{pmatrix}.$

### 1.3.2. Scalar multiplication

A scalar is a number.

If

$$A=\begin{pmatrix}a_{11} & a_{12} & \dots & a_{1n}\\ a_{21} & a_{22} & \dots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1} & a_{m2} & \dots & a_{mn}\end{pmatrix}$$

is an $m \times n$ matrix, and $k$ is a scalar, then

$$k \cdot A=\begin{pmatrix}k \cdot a_{11} & k \cdot a_{12} & \dots & k \cdot a_{1n}\\ k \cdot a_{21} & k \cdot a_{22} & \dots & k \cdot a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ k \cdot a_{m1} & k \cdot a_{m2} & \dots & k \cdot a_{mn}\end{pmatrix}.$$

*Properties*

Let $k$ and $p$ be scalars.

Distributive property of scalar multiplication over matrix addition

$$k(A+B)=kA+kB.$$

Distributive property of multiplication over addition

$$(k+p)A=kA+pA.$$

Associative property of scalar multiplication

$$(kp)A=k(pA);$$

$$1 \cdot A=A;$$

$$0 \cdot A=zero\ matrix.$$

*Example 1*

Let $C = \begin{pmatrix} 1 & -7 & 3 \\ 12 & 4 & -9 \end{pmatrix}$ and $k = 2$, then calculate $kC$.

*Solution*

$$2C = \begin{pmatrix} 2\cdot 1 & 2\cdot(-7) & 2\cdot 3 \\ 2\cdot 12 & 2\cdot 4 & 2\cdot(-9) \end{pmatrix} = \begin{pmatrix} 2 & -14 & 6 \\ 24 & 8 & -18 \end{pmatrix}$$

*Example 2*

Let $= \begin{pmatrix} 1 & -3 & 2 \\ 0 & 5 & 7 \\ -4 & 7 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 2 & -1 \\ 11 & 5 & 0 \\ 1 & -4 & 1 \end{pmatrix}$. Find $C = 2A - 3B$

*Solution*

$$2A = 2\cdot\begin{pmatrix} 1 & -3 & 2 \\ 0 & 5 & 7 \\ -4 & 7 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -6 & 4 \\ 0 & 10 & 14 \\ -8 & 14 & 6 \end{pmatrix};$$

$$3B = 3\cdot\begin{pmatrix} 4 & 2 & -1 \\ 11 & 5 & 0 \\ 1 & -4 & 1 \end{pmatrix} = \begin{pmatrix} 12 & 6 & -3 \\ 33 & 15 & 0 \\ 3 & -12 & 3 \end{pmatrix};$$

$$C = 2A - 3B = \begin{pmatrix} 2 & -6 & 4 \\ 0 & 10 & 14 \\ -8 & 14 & 6 \end{pmatrix} - \begin{pmatrix} 12 & 6 & -3 \\ 33 & 15 & 0 \\ 3 & -12 & 3 \end{pmatrix} = \begin{pmatrix} -10 & -12 & 7 \\ -33 & -5 & 14 \\ -11 & 26 & 3 \end{pmatrix}.$$

**1.3.3. Matrix multiplication**

Matrix multiplication is defined only when the number of columns in the first matrix is equal to the number of rows in the second matrix.

If

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{33} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}.$$

Then, $AB$ is calculated by multiplying rows of A with columns of B:

$$A = 4 \times 3, B = 3 \times 2;$$

$$AB = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$$
$$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \\ a_{41}b_{11} + a_{42}b_{21} + a_{43}b_{31} & a_{41}b_{12} + a_{42}b_{22} + a_{43}b_{32} \end{pmatrix}$$

*Properties of matrix multiplication*

Matrix multiplication is not commutative in general:

$A \cdot B \neq B \cdot A.$

Associative property:

$$(AB)C = A(BC).$$

Distributive property of matrix multiplication over addition

$(A + B)C = AC + BC,$ right distributive law,

or

$A(B + C) = AB + AC,$ left distributive property.

If $A$ is a square matrix, and $I$ is the identity matrix of the same order as $A$, then

$$IA = AI = A.$$

If $AB = BA$, then matrices $A$ and $B$ are called commutative matrices or commuting matrices.

*Example 1*

Compute $AB$ and $BA$, where $A = \begin{pmatrix} 1 & 3 \\ -2 & 1 \\ 0 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -1 & 5 & -7 & 1 \\ 0 & 5 & 0 & -6 & 2 \end{pmatrix}$.

*Solution 1*

$$A = 3 \times 2, B = 2 \times 5$$

$$AB = \begin{pmatrix} 1 & 3 \\ -2 & 1 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 3 & -1 & 5 & -7 & 1 \\ 0 & 5 & 0 & -6 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 \cdot 3 + 3 \cdot 0 & 1 \cdot (-1) + 3 \cdot 5 & 1 \cdot 5 + 3 \cdot 0 & 1 \cdot (-7) + 3 \cdot (-6) & 1 \cdot 1 + 3 \cdot 2 \\ -2 \cdot 3 + 1 \cdot 0 & -2 \cdot (-1) + 1 \cdot 5 & -2 \cdot 5 + 1 \cdot 0 & -2 \cdot (-7) + 1 \cdot (-6) & -2 \cdot 1 + 1 \cdot 2 \\ 0 \cdot 3 + 4 \cdot 0 & 0 \cdot (-1) + 4 \cdot 5 & 0 \cdot 5 + 4 \cdot 0 & 0 \cdot (-7) + 4 \cdot (-6) & 0 \cdot 1 + 4 \cdot 2 \end{pmatrix}$$
$$= \begin{pmatrix} 3 & 14 & 5 & -25 & 7 \\ -6 & 7 & -10 & 8 & 0 \\ 0 & 20 & 0 & -24 & 8 \end{pmatrix}$$

*Solution 2*

$$B = 2 \times 5, A = 3 \times 2$$

But to multiply $BA$, the number of columns in B (5) must equal the number of rows in A (3):

$$5 \neq 3.$$

Not compatible for multiplication.

*Example 2*

Let $A = \begin{pmatrix} 3 & 7 \\ -1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & -3 \\ 3 & 7 \end{pmatrix}$. Show that these matrices do not commute. That is, verify that $AB \neq BA$.

*Solution*

$$A = 2 \times 2, B = 2 \times 2$$

$$AB = \begin{pmatrix} 3 & 7 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 4 & -3 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} 3 \cdot 4 + 7 \cdot 3 & 3 \cdot (-3) + 7 \cdot 7 \\ (-1) \cdot 4 + 2 \cdot 3 & -1 \cdot (-3) + 2 \cdot 7 \end{pmatrix} = \begin{pmatrix} 33 & 40 \\ 2 & 17 \end{pmatrix}$$

$$BA = \begin{pmatrix} 4 & -3 \\ 3 & 7 \end{pmatrix} \begin{pmatrix} 3 & 7 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 4 \cdot 3 + (-3) \cdot (-1) & 4 \cdot 7 + (-3) \cdot 2 \\ 3 \cdot 3 + 7 \cdot (-1) & 3 \cdot 7 + 7 \cdot 2 \end{pmatrix} = \begin{pmatrix} 15 & 22 \\ 2 & 35 \end{pmatrix}$$

These matrices do not commute: $AB \neq BA$.

### 1.3.4. Powers of a matrix

If $A$ is an $n \times n$ matrix and if $k$ is a positive integer, then $A^k$ denotes the product of $k$ copies of $A$:

$$A^k = \underbrace{A \dots A}_{k}$$

$$A^0 = I$$

$$A^1 = A$$

$$A^2 = A \cdot A$$

$$A^3 = A \cdot A \cdot A$$

$$A^k = A \cdot A^{k-1}$$

*Example 1*

Find $A^3$ for $A = \begin{pmatrix} -1 & 0 \\ 7 & 3 \end{pmatrix}$.

*Solution*

$$A^3 = A \cdot A \cdot A = A^2 \cdot A$$

$$A^2 = \begin{pmatrix} -1 & 0 \\ 7 & 3 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 7 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 14 & 9 \end{pmatrix}$$

$$A^3 = A^2 \cdot A = \begin{pmatrix} 1 & 0 \\ 14 & 9 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 7 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 49 & 27 \end{pmatrix}$$

*Example 2*

Given the matrix

$$A = \begin{pmatrix} 0 & -2 & 1 \\ 5 & 6 & 3 \\ -2 & -1 & 4 \end{pmatrix}$$

compute the matrix $B = A^2 + 5E$, where $E$ is the identity matrix of the same size as $A$.

*Solution*

$$A^2 = \begin{pmatrix} 0 & -2 & 1 \\ 5 & 6 & 3 \\ -2 & -1 & 4 \end{pmatrix}\begin{pmatrix} 0 & -2 & 1 \\ 5 & 6 & 3 \\ -2 & -1 & 4 \end{pmatrix} =$$

$$\begin{pmatrix} -10-2 & -12-1 & -6+4 \\ 30-6 & -10+36-3 & 5+18+12 \\ -5-8 & 4-6-4 & -2-3+16 \end{pmatrix} = \begin{pmatrix} -12 & -13 & -2 \\ 24 & 23 & 35 \\ -13 & -6 & 11 \end{pmatrix}$$

$$B = A^2 + 5E = \begin{pmatrix} -12 & -13 & -2 \\ 24 & 23 & 35 \\ -13 & -6 & 11 \end{pmatrix} + 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -12 & -13 & -2 \\ 24 & 23 & 35 \\ -13 & -6 & 11 \end{pmatrix} + \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$
$$= \begin{pmatrix} -7 & -13 & -2 \\ 24 & 28 & 35 \\ -13 & -6 & 16 \end{pmatrix}$$

*Example 3*

Let matrix

$$C = \begin{pmatrix} 1 & -7 \\ 2 & 3 \end{pmatrix}$$

and polynomial

$$P(x) = x^3 + 4x - 6$$

Compute $P(C) = C^3 + 4C - 6E$, where $E$ is the identity matrix of the same size as $C$.

*Solution*

$$C^3 = C^2 \cdot C$$

$$C^2 = \begin{pmatrix} 1 & -7 \\ 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & -7 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1-14 & -7-21 \\ 2+6 & -14+9 \end{pmatrix} = \begin{pmatrix} -13 & -28 \\ 8 & -5 \end{pmatrix}$$

$$C^3 = C^2 \cdot C = \begin{pmatrix} -13 & -28 \\ 8 & -5 \end{pmatrix} \cdot \begin{pmatrix} 1 & -7 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} -13-56 & 91-84 \\ 8-10 & -56-15 \end{pmatrix} = \begin{pmatrix} -69 & 7 \\ -2 & -71 \end{pmatrix}$$

$$4C = 4 \cdot \begin{pmatrix} 1 & -7 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 4 & -28 \\ 8 & 12 \end{pmatrix}$$

$$6E = 6 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$$

$$P(C) = C^3 + 4C - 6E = \begin{pmatrix} -69 & 7 \\ -2 & -71 \end{pmatrix} + \begin{pmatrix} 4 & -28 \\ 8 & 12 \end{pmatrix} - \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$$
$$= \begin{pmatrix} -69+4-6 & 7-28-0 \\ -2+8-0 & -71+12-6 \end{pmatrix} = \begin{pmatrix} -71 & -21 \\ 6 & -65 \end{pmatrix}$$

### 1.3.5. The transpose of a matrix

Given an $m \times n$ matrix $A$, the transpose of $A$ is the $n \times m$ matrix, denoted by $A^T$, whose columns are formed from the corresponding rows of $A$.

*Properties of transposition*

$$(A^T)^T = A$$

$$(A + B)^T = A^T + B^T$$

$$(A \cdot B)^T = B^T \cdot A^T$$

A square matrix $A$ is called symmetric if it is equal to its transpose, that is, $A = A^T$.

*Example*

Let

$$B = \begin{pmatrix} 1 & -3 \\ 12 & 7 \\ 9 & -2 \end{pmatrix}.$$

*Solution*

To find the transpose of matrix $B$ we switch its rows with columns. That is, the first row becomes the first column, the second row becomes the second column, and the third row becomes the third column.

So, the transpose of $B$, denoted by $B^T$, is

$$B^T = \begin{pmatrix} 1 & 12 & 9 \\ -3 & 7 & -2 \end{pmatrix}.$$

## 1.4. Determinants: Basic concepts

For a square matrix $A$ of order $n$, we introduce a numerical characteristic called the determinant.

Determinant notation: $\det A$, or $|A|$, or $\Delta$.

1. If $n = 1$, then $A = (a_{11})$, and $\det A = |a_{11}| = a_{11}$.

2. If $n = 2$, then $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, and $\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$.
3. If $n = 3$, then $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, and $\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \cdot a_{22} \cdot a_{33} + a_{12} \cdot a_{23} \cdot a_{31} + a_{13} \cdot a_{21} \cdot a_{32} - a_{13} \cdot a_{22} \cdot a_{31} - a_{11} \cdot a_{23} \cdot a_{32} - a_{12} \cdot a_{21} \cdot a_{33}$.

This rule for calculating the determinant of a third-order matrix is called Sarrus' rule or the rule of triangles.

*Example 1*

Compute the determinant of

$$A = \begin{pmatrix} 4 & -1 \\ 3 & 2 \end{pmatrix}.$$

*Solution*

To compute the determinant of a $2 \times 2$ matrix, we use the formula $\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$.

$$\det A = \begin{vmatrix} 4 & -1 \\ 3 & 2 \end{vmatrix} = 4 \cdot 2 - (-1) \cdot 3 = 8 + 3 = 11$$

*Example 2*

Compute the determinant of

$$A = \begin{pmatrix} 4 & -1 & 3 \\ 2 & 5 & -2 \\ 0 & 7 & 2 \end{pmatrix}.$$

*Solution*

To compute the determinant of a $3 \times 3$ matrix, we use the formula $\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \cdot a_{22} \cdot a_{33} + a_{12} \cdot a_{23} \cdot a_{31} + a_{13} \cdot a_{21} \cdot a_{32} - a_{13} \cdot a_{22} \cdot a_{31} - a_{11} \cdot a_{23} \cdot a_{32} - a_{12} \cdot a_{21} \cdot a_{33}$

$$\begin{aligned} \det A = \begin{vmatrix} 4 & -1 & 3 \\ 2 & 5 & -2 \\ 0 & 7 & 2 \end{vmatrix} &\\ &= 4 \cdot 5 \cdot 2 + (-1) \cdot (-2) \cdot 0 + 3 \cdot 2 \cdot 7 - 3 \cdot 5 \cdot 0 - 4 \cdot (-2) \cdot 7 - (-1) \cdot 2 \cdot 2 \\ &= 40 + 0 + 42 - 0 + 56 + 4 = 142. \end{aligned}$$

*Properties of determinants*

1. The determinant of a matrix does not change when the matrix is transposed.
   If $A = \begin{pmatrix} 3 & 4 \\ -1 & 2 \end{pmatrix}$, then $A^T = \begin{pmatrix} 3 & -1 \\ 4 & 2 \end{pmatrix}$
   $$\det A = \begin{vmatrix} 3 & 4 \\ -1 & 2 \end{vmatrix} = 6 + 4 = 10; \ \det A^T = \begin{vmatrix} 3 & -1 \\ 4 & 2 \end{vmatrix} = 6 + 4 = 10$$

$$\det A = \det A^T.$$

The rows and columns of a determinant are called its rows and columns, respectively.

2. Interchanging two rows or two columns of a determinant changes its sign.
$A_1 = \begin{pmatrix} 1 & 0 & 3 \\ -2 & 1 & 1 \\ 4 & 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} -2 & 1 & 1 \\ 1 & 0 & 3 \\ 4 & 0 & 1 \end{pmatrix}$

$$\begin{vmatrix} 1 & 0 & 3 \\ -2 & 1 & 1 \\ 4 & 0 & 1 \end{vmatrix} = -\begin{vmatrix} -2 & 1 & 1 \\ 1 & 0 & 3 \\ 4 & 0 & 1 \end{vmatrix}$$
$$\det A_1 = -11;\ \det A_2 = 11$$

3. A determinant with two identical rows or columns is equal to zero.
If $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -4 & 1 & 7 \end{pmatrix}$, then $\det A = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -4 & 1 & 7 \end{vmatrix} = 0.$

4. If any row or column of a determinant is multiplied by a scalar, the value of the determinant is also multiplied by that scalar.
$$\begin{vmatrix} 4 & 1 & 3 \\ -6 & 3 & 0 \\ 8 & 1 & 7 \end{vmatrix} = \begin{vmatrix} 2 \cdot 2 & 1 & 3 \\ 2 \cdot (-3) & 3 & 0 \\ 2 \cdot 4 & 1 & 7 \end{vmatrix} = 2 \cdot \begin{vmatrix} 2 & 1 & 3 \\ -3 & 3 & 0 \\ 4 & 1 & 7 \end{vmatrix}$$

5. If all elements of any row or column of a determinant are zero, then the determinant is equal to zero.
$$\begin{vmatrix} 1 & 0 & -3 \\ 2 & 0 & 4 \\ -7 & 0 & 3 \end{vmatrix} = 0$$

6. If the elements of any row or column of a determinant are sums of two terms, then the determinant can be expressed as the sum of two corresponding determinants.
$$\begin{vmatrix} 1 & 4 & 11 \\ -3 & 5 & 7 \\ 2 & -6 & 20 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 5+6 \\ -3 & 5 & 4+3 \\ 2 & -6 & 10+10 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 5 \\ -3 & 5 & 4 \\ 2 & -6 & 10 \end{vmatrix} + \begin{vmatrix} 1 & 4 & 6 \\ -3 & 5 & 3 \\ 2 & -6 & 10 \end{vmatrix}$$

7. If the elements of one row (or column) are added to the corresponding elements of a parallel row (or column) multiplied by a scalar, the value of the determinant does not change.

$$\begin{vmatrix} 1 & 4 & 11 \\ -3 & 5 & 7 \\ 2 & -6 & 20 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 11 + 4 \cdot 2 \\ -3 & 5 & 7 + 5 \cdot 2 \\ 2 & -6 & 20 + (-6) \cdot 2 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 4 & 11 \\ -3 & 5 & 7 \\ 2 & -6 & 20 \end{vmatrix} = 526, \begin{vmatrix} 1 & 4 & 11 + 4 \cdot 2 \\ -3 & 5 & 7 + 5 \cdot 2 \\ 2 & -6 & 20 + (-6) \cdot 2 \end{vmatrix} = 526$$

This confirms that adding a multiple of one column to another does not change the determinant.

The minor of an element $a_{ij}$ of a determinant of order $n$ is the determinant of order $n-1$ obtained by deleting the row and the column that intersect at the position of this element.

Notation: $M_{ij}$

The cofactor of an element $a_{ij}$ of a determinant is the minor $M_{ij}$ taken with a plus sign if the $i+j$ is even, and with a minus sign if the sum $i+j$ is odd.

Notation: $A_{ij}$

$$A_{ij} = (-1)^{i+j} \cdot M_{ij}$$

For the element $a_{32}$ of the determinant $\begin{vmatrix} 1 & -2 & 3 \\ 2 & 3 & 0 \\ 4 & -1 & 2 \end{vmatrix}$, we will find its minor and cofactor.

The element $a_{32}$ is in the third row and second column. To compute $M_{32}$, remove the third row and second column:

$$M_{32} = \begin{vmatrix} 1 & -2 & 3 \\ 2 & 3 & 0 \\ 4 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = 0 - 6 = -6$$

$$A_{32} = (-1)^{3+2} M_{32} = (-1)^5 \cdot (-6) = 6.$$

8. The determinant is equal to the sum of the products of the elements of any row (or column) and their corresponding cofactors. For example, the expansion of the determinant along the $i$-th row has the following form:

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = a_{i1}A_{i1} + a_{i2}A_{i2} + \dots + a_{in}A_{in}.$$

Compute the determinant $\begin{vmatrix} 4 & 3 & -1 \\ 0 & 2 & 3 \\ 1 & 5 & 7 \end{vmatrix}$. Expand the determinant along the first row:

$$\begin{vmatrix} 4 & 3 & -1 \\ 0 & 2 & 3 \\ 1 & 5 & 7 \end{vmatrix} = 4 \cdot (-1)^{1+1} \begin{vmatrix} 2 & 3 \\ 5 & 7 \end{vmatrix} + 3 \cdot (-1)^{1+2} \begin{vmatrix} 0 & 3 \\ 1 & 7 \end{vmatrix} + (-1) \cdot (-1)^{1+3} \begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix}$$
$$= 4 \cdot (14 - 15) - 3 \cdot (0 - 3) + (-1) \cdot (0 - 2) = -4 + 9 + 2 = 7.$$

9. The sum of the products of the elements of any row (or column) of a determinant and the cofactors of the corresponding elements of a parallel row (or column) is equal to zero.

$$a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 0$$

10. The determinant of the product of two square matrices is equal to the product of their determinants.

$$|A \cdot B| = |A| \cdot |B|$$

If $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -1 \\ 4 & 5 \end{pmatrix}$, then $A \cdot B = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 11 & 9 \\ 24 & 11 \end{pmatrix}$

$$|A \cdot B| = \begin{vmatrix} 11 & 9 \\ 24 & 11 \end{vmatrix} = 121 - 216 = -95$$

$$|A| = \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = 3 - 8 = -5,\ |B| = \begin{vmatrix} 3 & -1 \\ 4 & 5 \end{vmatrix} = 15 + 4 = 19$$

$$|A| \cdot |B| = -5 \cdot 19 = -95$$

11. The determinant of a triangular matrix is equal to the product of the elements on its main diagonal.

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{vmatrix} = a_{11} \cdot a_{22} \cdot \dots \cdot a_{nn}$$

$$\begin{vmatrix} 3 & -10 & 2 \\ 0 & -2 & 70 \\ 0 & 0 & 11 \end{vmatrix} = 3 \cdot (-2) \cdot 11 = -66$$

12. The determinant of a diagonal matrix is equal to the product of the elements on its main diagonal.

$$\begin{vmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{vmatrix} = a_{11} \cdot a_{22} \cdot \dots \cdot a_{nn}$$

$$\begin{vmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 11 \end{vmatrix} = 3 \cdot (-2) \cdot 11 = -66$$

## 1.5. Inverse matrix

Let $A$ be a square matrix of order $n$:

$$A_{n\times n} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}.$$

A square matrix $A$ is called non-singular if the determinant of $A$ is not equal to zero, i.e.

$$\det A \neq 0.$$

If the matrix $A$ is non-singular, then there exists a unique matrix $A^{-1}$, such that

$$A \cdot A^{-1} = A^{-1} \cdot A = I,$$

where $I$ is the identity matrix.

The matrix $A^{-1}$ is called the inverse matrix.

Theorem on the existence and uniqueness of the inverse matrix: Every non-singular matrix $A$ has a unique inverse matrix.

*Properties of an inverse matrix*

$$\det(A^{-1}) = \frac{1}{\det A}$$

$$(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$$

$$(A^{-1})^T = (A^T)^{-1}$$

$$(A^{-1})^{-1} = A$$

*Algorithm for finding the inverse matrix.*

1. Calculate the determinant of the given square matrix $A$.
   If $\det A = 0$, then matrix $A$ is singular, and the inverse matrix does not exist.
   If $\det A \neq 0$, then a unique inverse matrix $A^{-1}$exists.
2. Calculate the inverse matrix using the formula:

$$A^{-1} = \frac{1}{det\ A}(A^*)^T,$$

where $A^*$ is the matrix of cofactors of $A$, $(A^*)^T$ is the adjugate (adjoint) matrix – the transpose of the cofactor matrix.

*Example*

Let $A = \begin{pmatrix} 4 & -2 & 3 \\ 1 & 0 & 7 \\ -1 & 2 & 4 \end{pmatrix}$, compute $A^{-1}$.

*Solution*

Let us check the determinant of matrix $A$:

$$\det A = \begin{vmatrix} 4 & -2 & 3 \\ 1 & 0 & 7 \\ -1 & 2 & 4 \end{vmatrix} = 4 \cdot (-1)^2 \begin{vmatrix} 0 & 7 \\ 2 & 4 \end{vmatrix} + (-2) \cdot (-1)^3 \begin{vmatrix} 1 & 7 \\ -1 & 4 \end{vmatrix} + 3 \cdot (-1)^4 \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix}$$
$$= 4 \cdot (-14) + 2 \cdot 11 + 3 \cdot 2 = -56 + 22 + 6 = -28$$

$$\det A \neq 0$$

If $\det A \neq 0$, then a unique inverse matrix $A^{-1}$exists.

$$A^* = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

| | | |
|---|---|---|
| $A_{11} = (-1)^{1+1}\begin{vmatrix}0 & 7\\ 2 & 4\end{vmatrix} = -14$ | $A_{21} = (-1)^{2+1}\begin{vmatrix}-2 & 3\\ 2 & 4\end{vmatrix} = 14$ | $A_{31} = (-1)^{3+1}\begin{vmatrix}-2 & 3\\ 0 & 7\end{vmatrix} = -14$ |
| $A_{12} = (-1)^{1+2}\begin{vmatrix}1 & 7\\ -1 & 4\end{vmatrix} = -11$ | $A_{22} = (-1)^{2+2}\begin{vmatrix}4 & 3\\ -1 & 4\end{vmatrix} = 19$ | $A_{32} = (-1)^{3+2}\begin{vmatrix}4 & 3\\ 1 & 7\end{vmatrix} = -25$ |
| $A_{13} = (-1)^{1+3}\begin{vmatrix}1 & 0\\ -1 & 2\end{vmatrix} = 2$ | $A_{23} = (-1)^{2+3}\begin{vmatrix}4 & -2\\ -1 & 2\end{vmatrix} = -6$ | $A_{33} = (-1)^{3+3}\begin{vmatrix}4 & -2\\ 1 & 0\end{vmatrix} = 2$ |

$$A^* = \begin{pmatrix}-14 & -11 & 2\\ 14 & 19 & -6\\ -14 & -25 & 2\end{pmatrix}$$

Transpose the cofactor matrix (adjugate):

$$(A^*)^T = \begin{pmatrix}-14 & 14 & -14\\ -11 & 19 & -25\\ 2 & -6 & 2\end{pmatrix}$$

$$A^{-1} = \frac{1}{-28}\begin{pmatrix}-14 & 14 & -14\\ -11 & 19 & -25\\ 2 & -6 & 2\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & -\frac{1}{2} & \frac{1}{2}\\ \frac{11}{28} & -\frac{19}{28} & \frac{25}{28}\\ -\frac{1}{14} & \frac{3}{14} & -\frac{1}{14}\end{pmatrix}$$

*Verification*

$$A \cdot A^{-1} = \begin{pmatrix}4 & -2 & 3\\ 1 & 0 & 7\\ -1 & 2 & 4\end{pmatrix}\begin{pmatrix}\frac{1}{2} & -\frac{1}{2} & \frac{1}{2}\\ \frac{11}{28} & -\frac{19}{28} & \frac{25}{28}\\ -\frac{1}{14} & \frac{3}{14} & -\frac{1}{14}\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

### 1.6. Matrix equations

*Equation of the form $A \cdot X = B$.*

Let matrices $A$ and $B$ be given. Assume that matrix $A$ is non-singular (invertible). Multiply the matrix equation on the left by $A^{-1}$:

$$A^{-1} \cdot A \cdot X = A^{-1} \cdot B.$$

Since $A^{-1} \cdot A = I$ (the identity matrix), we obtain

$$I \cdot X = A^{-1} \cdot B$$

$$X = A^{-1} \cdot B.$$

The solution to the matrix equation $A \cdot X = B$ is given by:

$$X = A^{-1} \cdot B.$$

*Example*

Solve the matrix equation $A \cdot X = B$,

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & -1 & 2 \\ 4 & 3 & -1 \end{pmatrix}, B = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

*Solution*

The solution to the matrix equation $A \cdot X = B$ is given by

$$X = A^{-1} \cdot B.$$

Let us find the inverse matrix $A^{-1}$ of the matrix $A$:

$$\det A = \begin{vmatrix} 3 & 2 & 1 \\ 2 & -1 & 2 \\ 4 & 3 & -1 \end{vmatrix} = (-1)^2 \cdot 3 \cdot \begin{vmatrix} -1 & 2 \\ 3 & -1 \end{vmatrix} + (-1)^3 \cdot 2 \cdot \begin{vmatrix} 2 & 2 \\ 4 & -1 \end{vmatrix} + (-1)^4 \cdot 1 \cdot \begin{vmatrix} 2 & -1 \\ 4 & 3 \end{vmatrix}$$
$$= 3 \cdot (-5) - 2 \cdot (-10) + 2 = -15 + 20 + 10 = 15 \neq 0.$$

Since $|A| \neq 0$, the matrix $A$ is non-singular. Let us construct the adjugate matrix:

$$A^* = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

| | | |
|---|---|---|
| $A_{11} = (-1)^{1+1} \begin{vmatrix} -1 & 2 \\ 3 & -1 \end{vmatrix} = -5$ | $A_{21} = (-1)^{2+1} \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} = 5$ | $A_{31} = (-1)^{3+1} \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5$ |
| $A_{12} = (-1)^{1+2} \begin{vmatrix} 2 & 2 \\ 4 & -1 \end{vmatrix} = 10$ | $A_{22} = (-1)^{2+2} \begin{vmatrix} 3 & 1 \\ 4 & -1 \end{vmatrix} = -7$ | $A_{32} = (-1)^{3+2} \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} = -4$ |
| $A_{13} = (-1)^{1+3} \begin{vmatrix} 2 & -1 \\ 4 & 3 \end{vmatrix} = 10$ | $A_{23} = (-1)^{2+3} \begin{vmatrix} 3 & 2 \\ 4 & 3 \end{vmatrix} = -1$ | $A_{33} = (-1)^{3+3} \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} = -7$ |

$$A^* = \begin{pmatrix} -5 & 10 & 10 \\ 5 & -7 & -1 \\ 5 & -4 & -7 \end{pmatrix}.$$

Transpose the cofactor matrix (adjugate):

$$(A^*)^T = \begin{pmatrix} -5 & 5 & 5 \\ 10 & -7 & -4 \\ 10 & -1 & -7 \end{pmatrix}$$

$$A^{-1} = \frac{1}{15}\begin{pmatrix} -5 & 5 & 5 \\ 10 & -7 & -4 \\ 10 & -1 & -7 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{7}{15} & -\frac{4}{15} \\ \frac{2}{3} & -\frac{1}{15} & -\frac{7}{15} \end{pmatrix}$$

$$X = A^{-1} \cdot B = \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{7}{15} & -\frac{4}{15} \\ \frac{2}{3} & -\frac{1}{15} & -\frac{7}{15} \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \cdot 2 + \frac{1}{3} \cdot (-2) + \frac{1}{3} \cdot 1 \\ \frac{2}{3} \cdot 2 + \left(-\frac{7}{15}\right) \cdot (-2) + \left(-\frac{4}{15}\right) \cdot 1 \\ \frac{2}{3} \cdot 2 + \left(-\frac{1}{15}\right) \cdot (-2) + \left(-\frac{7}{15}\right) \cdot 1 \end{pmatrix} =$$

$$\begin{pmatrix} -\frac{2}{3} - \frac{2}{3} + \frac{1}{3} \\ \frac{4}{3} + \frac{14}{15} - \frac{4}{15} \\ \frac{4}{3} + \frac{2}{15} - \frac{7}{15} \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.$$

The solution of the matrix equation is a column matrix:

$$X = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.$$

*Equation of the form* $X \cdot A = B$.

Let matrices $A$ and $B$ be given. Assume that matrix $A$ is non-singular (invertible). Multiply the matrix equation on the right by $A^{-1}$:

$$X \cdot A \cdot A^{-1} = B \cdot A^{-1}.$$

Since $A \cdot A^{-1} = I$ (the identity matrix), we obtain

$$X \cdot I = B \cdot A^{-1}$$

$$X = B \cdot A^{-1}.$$

The solution to the matrix equation $X \cdot A = B$ is given by

$$X = B \cdot A^{-1}.$$

*Example*

Solve the matrix equation $X \cdot A = B$:

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 3 & 4 & 7 \\ -1 & 0 & 5 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 7 & 9 \\ 5 & -1 & 0 \end{pmatrix}.$$

*Solution*

The solution to the matrix equation $X \cdot A = B$ is given by

$$X = B \cdot A^{-1}.$$

Let us find the inverse matrix $A^{-1}$ of the matrix $A$.

$$\det A = \begin{vmatrix} 1 & -1 & 0 \\ 3 & 4 & 7 \\ -1 & 0 & 5 \end{vmatrix} = (-1)^2 \cdot 1 \cdot \begin{vmatrix} 4 & 7 \\ 0 & 5 \end{vmatrix} + (-1)^3 \cdot (-1) \cdot \begin{vmatrix} 3 & 7 \\ -1 & 5 \end{vmatrix} + 0 = 20 + 22 = 42 \neq 0$$

Since $|A| \neq 0$, the matrix $A$ is non-singular. Let us construct the adjugate matrix:

$$A^* = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

| | | |
|---|---|---|
| $A_{11} = (-1)^{1+1} \begin{vmatrix} 4 & 7 \\ 0 & 5 \end{vmatrix} = 20$ | $A_{21} = (-1)^{2+1} \begin{vmatrix} -1 & 0 \\ 0 & 5 \end{vmatrix} = 5$ | $A_{31} = (-1)^{3+1} \begin{vmatrix} -1 & 0 \\ 4 & 7 \end{vmatrix} = -7$ |
| $A_{12} = (-1)^{1+2} \begin{vmatrix} 3 & 7 \\ -1 & 5 \end{vmatrix} = -22$ | $A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 0 \\ -1 & 5 \end{vmatrix} = 5$ | $A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 0 \\ 3 & 7 \end{vmatrix} = -7$ |
| $A_{13} = (-1)^{1+3} \begin{vmatrix} 3 & 4 \\ -1 & 0 \end{vmatrix} = 4$ | $A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & -1 \\ -1 & 0 \end{vmatrix} = 1$ | $A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & -1 \\ 3 & 4 \end{vmatrix} = 7$ |

$$A^* = \begin{pmatrix} 20 & -22 & 4 \\ 5 & 5 & 1 \\ -7 & -7 & 7 \end{pmatrix}.$$

Transpose the cofactor matrix (adjugate):

$$(A^*)^T = \begin{pmatrix} 20 & 5 & -7 \\ -22 & 5 & -7 \\ 4 & 1 & 7 \end{pmatrix}$$

$$A^{-1} = \frac{1}{42}\begin{pmatrix} 20 & -5 & -7 \\ -22 & 5 & -7 \\ 4 & 1 & 7 \end{pmatrix} = \begin{pmatrix} \frac{10}{21} & \frac{5}{42} & -\frac{1}{6} \\ -\frac{11}{21} & \frac{5}{42} & -\frac{1}{6} \\ \frac{2}{21} & \frac{1}{42} & \frac{1}{6} \end{pmatrix}$$

$$X = A^{-1} \cdot B = \begin{pmatrix} \frac{10}{21} & \frac{5}{42} & -\frac{1}{6} \\ -\frac{11}{21} & \frac{5}{42} & -\frac{1}{6} \\ \frac{2}{21} & \frac{1}{42} & \frac{1}{6} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -2 \\ 0 & 7 & 9 \\ 5 & -1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{10}{21} - \frac{5}{6} & \frac{35}{42} + \frac{1}{6} & -\frac{20}{21} + \frac{45}{42} \\ -\frac{11}{21} - \frac{5}{6} & \frac{35}{42} + \frac{1}{6} & \frac{22}{21} + \frac{45}{42} \\ \frac{2}{21} + \frac{5}{6} & \frac{7}{42} - \frac{1}{6} & -\frac{4}{21} + \frac{9}{42} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{5}{14} & 1 & \frac{5}{42} \\ -\frac{19}{14} & 1 & \frac{89}{42} \\ \frac{13}{14} & 0 & \frac{1}{42} \end{pmatrix}.$$

The solution of the matrix equation is a matrix

$$X = \begin{pmatrix} -\frac{5}{14} & 1 & \frac{5}{42} \\ -\frac{19}{14} & 1 & \frac{89}{42} \\ \frac{13}{14} & 0 & \frac{1}{42} \end{pmatrix}.$$

*Equation of the form $A \cdot X \cdot B = C$.*

Let matrices $A, B$, and $C$ be given. Assume that matrices $A$ and $B$ are non-singular (invertible). Multiply the matrix equation on the left by $A^{-1}$ and on the right by $B^{-1}$:

$$A^{-1} \cdot A \cdot X \cdot B \cdot B^{-1} = A^{-1} \cdot C \cdot B^{-1}$$

$$I \cdot X \cdot I = A^{-1} \cdot C \cdot B^{-1}$$

$X = A^{-1} \cdot C \cdot B^{-1}$.

The solution to the matrix equation $A \cdot X \cdot B = C$ is given by

$$X = A^{-1} \cdot C \cdot B^{-1}$$

*Example*

Solve the matrix equation $A \cdot X \cdot B = C$,

$$A = \begin{pmatrix} -3 & 0 & 1 \\ 8 & -1 & 0 \\ 3 & -2 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & -2 & 1 \\ 3 & -7 & 1 \\ 4 & 0 & -3 \end{pmatrix}, C = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

*Solution*

The solution to the matrix equation $A \cdot X \cdot B = C$ is given by

$$X = A^{-1} \cdot C \cdot B^{-1}.$$

Let us find the inverse matrix $A^{-1}$ of the matrix $A$:

$\det A = \begin{vmatrix} -3 & 0 & 1 \\ 8 & -1 & 0 \\ 3 & -2 & 1 \end{vmatrix} = (-1)^2 \cdot (-3) \cdot \begin{vmatrix} -1 & 0 \\ -2 & 1 \end{vmatrix} + 0 + (-1)^4 \cdot 1 \cdot \begin{vmatrix} 8 & -1 \\ 3 & -2 \end{vmatrix} = -3 \cdot (-1) + (-13) = -10 \neq 0.$

Since $|A| \neq 0$, the matrix $A$ is non-singular. Let us construct the adjugate matrix:

$$A^* = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

| | | |
|---|---|---|
| $A_{11} = (-1)^{1+1} \begin{vmatrix} -1 & 0 \\ -2 & 1 \end{vmatrix} = -1$ | $A_{21} = (-1)^{2+1} \begin{vmatrix} 0 & 1 \\ -2 & 1 \end{vmatrix} = -2$ | $A_{31} = (-1)^{3+1} \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} = 1$ |

| | | |
|---|---|---|
| $A_{12} = (-1)^{1+2}\begin{vmatrix} 8 & 0 \\ 3 & 1 \end{vmatrix} = -8$ | $A_{22} = (-1)^{2+2}\begin{vmatrix} -3 & 1 \\ 3 & 1 \end{vmatrix} = -6$ | $A_{32} = (-1)^{3+2}\begin{vmatrix} -3 & 1 \\ 8 & 0 \end{vmatrix} = 8$ |
| $A_{13} = (-1)^{1+3}\begin{vmatrix} 8 & -1 \\ 3 & -2 \end{vmatrix} = -13$ | $A_{23} = (-1)^{2+3}\begin{vmatrix} -3 & 0 \\ 3 & -2 \end{vmatrix} = -6$ | $A_{33} = (-1)^{3+3}\begin{vmatrix} -3 & 0 \\ 8 & -1 \end{vmatrix} = 3$ |

$$A^* = \begin{pmatrix} -1 & -8 & -13 \\ -2 & -6 & -6 \\ 1 & 8 & 3 \end{pmatrix}.$$

Transpose the cofactor matrix (adjugate):

$$(A^*)^T = \begin{pmatrix} -1 & -2 & 1 \\ -8 & -6 & 8 \\ -13 & -6 & 3 \end{pmatrix}$$

$$A^{-1} = \frac{1}{-10}\begin{pmatrix} -1 & -2 & 1 \\ -8 & -6 & 8 \\ -13 & -6 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{10} & \frac{1}{5} & -\frac{1}{10} \\ \frac{4}{5} & \frac{3}{5} & -\frac{4}{5} \\ \frac{13}{10} & \frac{3}{5} & -\frac{3}{10} \end{pmatrix}.$$

Let us find the inverse matrix $B^{-1}$ of the matrix $B$:

$$\det B = \begin{vmatrix} 0 & -2 & 1 \\ 3 & -7 & 1 \\ 4 & 0 & -3 \end{vmatrix} = 0 + (-1)^3 \cdot (-2) \cdot \begin{vmatrix} 3 & 1 \\ 4 & -3 \end{vmatrix} + (-1)^4 \cdot 1 \cdot \begin{vmatrix} 3 & -7 \\ 4 & 0 \end{vmatrix} = 2 \cdot (-13) + 28 = 2 \neq 0.$$

Since $|B| \neq 0$, the matrix $A$ is non-singular. Let us construct the adjugate matrix:

$$B^* = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix}$$

| | | |
|---|---|---|
| $B_{11} = (-1)^{1+1}\begin{vmatrix} -7 & 1 \\ 0 & -3 \end{vmatrix} = 21$ | $B_{21} = (-1)^{2+1}\begin{vmatrix} -2 & 1 \\ 0 & -3 \end{vmatrix} = -6$ | $B_{31} = (-1)^{3+1}\begin{vmatrix} -2 & 1 \\ -7 & 1 \end{vmatrix} = 5$ |
| $B_{12} = (-1)^{1+2}\begin{vmatrix} 3 & 1 \\ 4 & -3 \end{vmatrix} = 13$ | $B_{22} = (-1)^{2+2}\begin{vmatrix} 0 & 1 \\ 4 & -3 \end{vmatrix} = -4$ | $B_{32} = (-1)^{3+2}\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} = 3$ |
| $B_{13} = (-1)^{1+3}\begin{vmatrix} 3 & -7 \\ 4 & 0 \end{vmatrix} = 28$ | $B_{23} = (-1)^{2+3}\begin{vmatrix} 0 & -2 \\ 4 & 0 \end{vmatrix} = -8$ | $B_{33} = (-1)^{3+3}\begin{vmatrix} 0 & -2 \\ 3 & -7 \end{vmatrix} = 6$ |

$$B^* = \begin{pmatrix} 21 & 13 & 28 \\ -6 & -4 & -8 \\ 5 & 3 & 6 \end{pmatrix}.$$

Transpose the cofactor matrix (adjugate):

$$(B^*)^T = \begin{pmatrix} 21 & -6 & 5 \\ 13 & -4 & 3 \\ 28 & -8 & 6 \end{pmatrix}$$

$$B^{-1} = \frac{1}{2}\begin{pmatrix} 21 & -6 & 5 \\ 13 & -4 & 3 \\ 28 & -8 & 6 \end{pmatrix} = \begin{pmatrix} \frac{21}{2} & -3 & \frac{5}{2} \\ \frac{13}{2} & -2 & \frac{3}{2} \\ 14 & -4 & 3 \end{pmatrix}$$

$$X = A^{-1} \cdot C \cdot B^{-1} = \begin{pmatrix} \frac{1}{10} & \frac{1}{5} & -\frac{1}{10} \\ \frac{4}{5} & \frac{3}{5} & -\frac{4}{5} \\ \frac{13}{10} & \frac{3}{5} & -\frac{3}{10} \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{21}{2} & -3 & \frac{5}{2} \\ \frac{13}{2} & -2 & \frac{3}{2} \\ 14 & -4 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{10} + \left(-\frac{1}{10}\right) \\ -\frac{4}{5} + \left(-\frac{4}{5}\right) \\ -\frac{13}{10} + \left(-\frac{3}{10}\right) \end{pmatrix} \cdot \begin{pmatrix} \frac{21}{2} & -3 & \frac{5}{2} \\ \frac{13}{2} & -2 & \frac{3}{2} \\ 14 & -4 & 3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} \\ -\frac{8}{5} \\ -\frac{8}{5} \end{pmatrix} \cdot \begin{pmatrix} \frac{21}{2} & -3 & \frac{5}{2} \\ \frac{13}{2} & -2 & \frac{3}{2} \\ 14 & -4 & 3 \end{pmatrix}$$

$$= \left(-\frac{1}{5} \cdot \frac{21}{2} + \left(-\frac{8}{5}\right) \cdot \frac{13}{2} + \left(-\frac{8}{5}\right) \cdot 14 \quad -\frac{1}{5} \cdot (-3) + \left(-\frac{8}{5}\right) \cdot (-2) + \left(-\frac{8}{5}\right) \cdot (-4) \quad -\frac{1}{5} \cdot \frac{5}{2} + \left(-\frac{8}{5}\right) \cdot \frac{3}{2}\right.$$

$$= \begin{pmatrix} -\frac{349}{10} & \frac{51}{5} & -\frac{77}{10} \end{pmatrix}.$$

The solution of the matrix equation is a row matrix:

$$X = \begin{pmatrix} -\frac{349}{10} & \frac{51}{5} & -\frac{77}{10} \end{pmatrix}.$$

**1.7. Rank of a matrix**

Consider a matrix $A$ of dimensions $m \times n$:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

Let us select k rows and k columns in it, where $\mathrm{k} \le \min(\mathrm{m}, \mathrm{n})$.

The elements at the intersection of selected rows and columns form a square matrix of order k, whose determinant is called a minor of order k of the matrix $A$.
The rank of a matrix is the maximum order of the non-zero minors of matrix $A$.

*Notation:* $r$, or $r(A)$, or rank $A$.
Any non-zero minor of order r (where r is the rank of the matrix) is called a basic minor.

*Properties of the rank*

1. The rank of a matrix does not change under transposition.
2. If a zero row is removed from a matrix, its rank remains unchanged.
3. The rank of a matrix is invariant under elementary row operations.

Matrices $A$ and $B$ are said to be equivalent if $r(A) = r(B)$.
The notation for equivalent matrices is: $A \sim B$.

*Methods for determining the rank of a matrix*

1. Method of bordering minors

Suppose a non-zero minor of order k, denoted by M, is found in the matrix.
Now consider only those minors of order $(\text{k} + 1)$ that contain (i.e., border) the minor M.
If all such bordering minors are equal to zero, then the rank of the matrix is k.
Otherwise, if at least one bordering minor of order $(\text{k} + 1)$ is non-zero, the procedure is repeated with this new minor.

*Example*

Find the rank of the matrix using the method of bordering minors.

$$A = \begin{pmatrix} 1 & 2 & 1 & 3 & 4 \\ 3 & 4 & 2 & 6 & 8 \\ 1 & 2 & 1 & 3 & 4 \end{pmatrix}$$

Let us fix a non-zero minor of order 2:

$$M_2 = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0.$$

Let us find all minors of order 3 that border the minor $M_2$:

$$M_3 = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 4 & 2 \\ 1 & 2 & 1 \end{vmatrix} = (-1)^2 \cdot 1 \cdot \begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix} + (-1)^3 \cdot 2 \cdot \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} + (-1)^4 \cdot 1 \cdot \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 0 - 2 + 2 = 0$$

$$M_3 = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 6 \\ 1 & 2 & 3 \end{vmatrix} = (-1)^2 \cdot 1 \cdot \begin{vmatrix} 4 & 6 \\ 2 & 3 \end{vmatrix} + (-1)^3 \cdot 2 \cdot \begin{vmatrix} 3 & 6 \\ 1 & 3 \end{vmatrix} + (-1)^4 \cdot 3 \cdot \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 0 - 6 + 6 = 0$$

$$M_3 = \begin{vmatrix} 1 & 2 & 4 \\ 3 & 4 & 8 \\ 1 & 2 & 4 \end{vmatrix} = (-1)^2 \cdot 1 \cdot \begin{vmatrix} 4 & 8 \\ 2 & 4 \end{vmatrix} + (-1)^3 \cdot 2 \cdot \begin{vmatrix} 3 & 8 \\ 1 & 4 \end{vmatrix} + (-1)^4 \cdot 4 \cdot \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 0 - 8 + 8 = 0.$$

As all minors of order 3 that border $M_2$ are zero, the rank of the matrix is 2, that is, $r(A) = 2$.

The concept of the rank of a matrix is closely related to the concept of linear dependence (or independence) of its rows or columns.

Let us define the notions of dependence and independence for the rows of a matrix. These definitions apply similarly to the columns.

Let a matrix $A$ be given:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

Let us denote its rows as follows:

$$e_1 = (a_{11}\ a_{12} \dots a_{1n}), e_2 = (a_{21}\ a_{22}\ \dots a_{2n}), \dots, e_m = (a_{m1}\ a_{m2}\ \dots\ a_{mn}).$$

Two rows of a matrix are said to be equal if their corresponding elements are equal: $e_k = e_s, a_{kj} = a_{sj}\ j = \overline{1,n}$.

Arithmetic operations on the rows of a matrix (multiplying a row by a number, adding rows) are defined as operations performed element-wise:

$$\lambda e_k = (\lambda a_{k1}\ \lambda a_{k2}\ \dots\ \lambda a_{kn});$$

$$e_k + e_s = [(a_{k1} + a_{s1})\ (a_{k2} + a_{s2})\ \dots\ (a_{kn} + a_{sn})].$$

A row $e$ is said to be a linear combination of the rows $e_1, e_2, \dots, e_s$ of a matrix if there exist real numbers $\lambda_1, \lambda_2, \dots, \lambda_s$, such that $e = \lambda_1 e_1 + \lambda_2 e_2 + \dots + \lambda_s e_s$.

The rows $e_1, e_2, \dots, e_m$ of a matrix are said to be linearly dependent if there exist numbers $\lambda_1, \lambda_2, \dots, \lambda_m$, not all equal to zero, such that a linear combination of the rows is equal to the zero row:

$$\lambda_1 e_1 + \lambda_2 e_2 + \dots + \lambda_m e_m = 0,$$

where $0 = (0\ 0\ \dots 0)$.

Linear dependence of the rows of a matrix means that at least one row is a linear combination of the others.

If a linear combination of the rows $\lambda_1 e_1 + \lambda_2 e_2 + \dots + \lambda_m e_m = 0$ holds only when all the coefficients $\lambda_i$ are equal to zero, i.e. $\lambda_1 = \lambda_2, = \dots = \lambda_m = 0$, then the rows $e_1, e_2, \dots, e_m$ are said to be linearly independent.

*Theorem on the rank of a matrix*
The rank of a matrix is equal to the maximum number of its linearly independent rows or columns, through which all other rows (columns) can be expressed.

2. Method of elementary transformations

Using elementary transformations, a matrix can be brought to a form in which all its elements, except
$\mathrm{a}_{11}, \mathrm{a}_{22}, \dots, \mathrm{a}_{\mathrm{rr}}$ (where $\mathrm{r} \leq min(\mathrm{m}, \mathrm{n})$), are zero. The rank of the matrix is r.

*Example*

Find the rank of matrix A using the method of elementary transformations, if

$$A=\begin{pmatrix}1 & -2 & -4 & 3\\ 2 & 1 & -3 & 5\\ 1 & 8 & 6 & 1\end{pmatrix}.$$

*Solution*

Let us perform elementary transformations.

$$A=\begin{pmatrix}1 & -2 & -4 & 3\\ 2 & 1 & -3 & 5\\ 1 & 8 & 6 & 1\end{pmatrix}\underset{1.r\cdot(-1)+3.r}{\overset{1.r\cdot(-2)+2.r}{\sim}}\begin{pmatrix}1 & -2 & -4 & 3\\ 0 & 5 & 5 & -1\\ 0 & 10 & 10 & -2\end{pmatrix}\overset{2.r\cdot(-2)+3.r}{\sim}\begin{pmatrix}1 & -2 & -4 & 3\\ 0 & 5 & 5 & -1\\ 0 & 0 & 0 & 0\end{pmatrix}\sim$$

$$\begin{pmatrix}1 & -2 & -4 & 3\\ 0 & 5 & 5 & -1\end{pmatrix}$$

Since there are 2 non-zero rows, the rank of matrix A is 2.

*Properties of the rank of a matrix*

The rank of a matrix does not change under the following operations:

1) transposing the matrix;
2) swapping its columns or rows;
3) multiplying all elements of a column (or row) by a nonzero number;
4) adding to one of its columns (or rows) another column (or row) multiplied by a scalar;
5) deleting a column (or row) consisting entirely of zeros;
6) deleting a column (or row) that is a linear combination of other columns (or rows).

## 2. Systems of linear equations

A system of linear algebraic equations with m equations and n unknowns is a system of the form

$$\begin{cases} a_{11}\cdot x_1 + a_{12}\cdot x_2 + \ldots + a_{1n}\cdot x_n = b_1\\ a_{21}\cdot x_1 + a_{22}\cdot x_2 + \ldots + a_{2n}\cdot x_n = b_2\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ a_{m1}\cdot x_1 + a_{m2}\cdot x_2 + \ldots + a_{mn}\cdot x_n = b_m\end{cases},$$

where $a_{ij}$ are the coefficients of the system, $b_i$ are the constant terms, $x_j$ are the unknowns, $i=\overline{1,m}, j=\overline{1,n}$.

Let us write the system in matrix form.

$$A\cdot X = B,$$

where $A$ is the coefficient matrix, $X$ is the column vector of unknowns, and $B$ is the column vector of constants.

The augmented matrix of a system is the matrix of the form

$$\bar{A} = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right).$$

A solution of a system is a set of n values of the unknowns,

$$x_1 = c_1, x_2 = c_2, \dots, x_n = c_n,$$

such that, when substituted into the system, all equations become true equalities.

If all the free coefficients of the system are equal to zero ($b_m = 0$), then the system is called homogeneous. Otherwise, i.e., if at least one free coefficient is not equal to zero, the system is called non-homogeneous.

A system of equations is called consistent (compatible) if it has at least one solution.

A system of equations is called inconsistent (incompatible) if it has no solutions.

A consistent system is called determinate if it has a unique solution.

A consistent system is called indeterminate if it has more than one solution.

Each solution of an indeterminate system is called a particular solution of the system.

The set of all particular solutions is called the general solution of the system.

2.1. Solution of non-singular linear systems

Let a system of $n$ linear equations in n unknowns be given:

$$\begin{cases} a_{11} \cdot x_1 + a_{12} \cdot x_2 + \dots + a_{1n} \cdot x_n = b_1 \\ a_{21} \cdot x_1 + a_{22} \cdot x_2 + \dots + a_{2n} \cdot x_n = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1} \cdot x_1 + a_{n2} \cdot x_2 + \dots + a_{nn} \cdot x_n = b_n \end{cases},$$

or in matrix form $A \cdot X = B$.

$$\det A = \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{vmatrix}$$

If the determinant of the system is nonzero, the system is called non-singular.

*Matrix method of solution*

$$X = A^{-1} \cdot B$$

*Example*

Solve the system using the matrix method

$$\begin{cases} x + 2y + z = 5 \\ 2x + 2y + z = 6. \\ x + 2y + 3z = 9 \end{cases}$$

*Solution*

We find the solution of the system using the formula

$$X = A^{-1} \cdot B.$$

The coefficient matrix of the system is of the form

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

$$B = \begin{pmatrix} 5 \\ 6 \\ 9 \end{pmatrix}.$$

Let us find the inverse matrix $A^{-1}$ of the matrix $A$.

$$\det A = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{vmatrix} = (-1)^2 \cdot 1 \cdot \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + (-1)^3 \cdot 2 \cdot \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + (-1)^4 \cdot 1 \cdot \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix}$$
$$= 4 - 2 \cdot 5 + 2 = -4 \neq 0$$

Since $|A| \neq 0$, the matrix $A$ is non-singular. Let us construct the adjugate matrix:

$$A^* = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

| | | |
|---|---|---|
| $A_{11} = (-1)^{1+1} \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} = 4$ | $A_{21} = (-1)^{2+1} \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} = -4$ | $A_{31} = (-1)^{3+1} \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} = 0$ |
| $A_{12} = (-1)^{1+2} \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = -5$ | $A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = 2$ | $A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1$ |
| $A_{13} = (-1)^{1+3} \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = 2$ | $A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0$ | $A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} = -2$ |

$$A^* = \begin{pmatrix} 4 & -5 & 2 \\ -4 & 2 & 0 \\ 0 & 1 & -2 \end{pmatrix}.$$

Transpose the cofactor matrix (adjugate):

$$(A^*)^T = \begin{pmatrix} 4 & -4 & 0 \\ -5 & 2 & 1 \\ 2 & 0 & -2 \end{pmatrix}$$

$$A^{-1} = \frac{1}{-4}\begin{pmatrix} 4 & -4 & 0 \\ -5 & 2 & 1 \\ 2 & 0 & -2 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ \frac{5}{4} & -\frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

$$X = A^{-1} \cdot B = \begin{pmatrix} -1 & 1 & 0 \\ \frac{5}{4} & -\frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 6 \\ 9 \end{pmatrix} = \begin{pmatrix} -5+6 \\ \frac{25}{4} + (-3) + \left(-\frac{9}{4}\right) \\ -\frac{5}{2} + \frac{9}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$X = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

*Cramer's Rule*

Cramer's Rule is a method for solving a system of $n$ linear equations with $n$ unknowns using determinants.

For a system of 3 equations in 3 unknowns,

$$\begin{cases} a_{11} \cdot x_1 + a_{12} \cdot x_2 + a_{13} \cdot x_3 = b_1 \\ a_{21} \cdot x_1 + a_{22} \cdot x_2 + a_{23} \cdot x_3 = b_2 \\ a_{31} \cdot x_1 + a_{32} \cdot x_2 + a_{33} \cdot x_3 = b_3 \end{cases},$$

build the coefficient matrix $A$,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

$\Delta = det(A)$, the determinant of the coefficient matrix.

Each matrix is formed by replacing one column of matrix $A$ with the constants (right-hand side values): $A_{x_1}$– replace the 1st column of $A$ with $\vec{b}$; $A_{x_2}$– replace the 2st column of $A$ with $\vec{b}$; $A_{x_3}$– replace the 3st column of $A$ with $\vec{b}$.

$$x_1 = \frac{\Delta_{x_1}}{\Delta}; x_2 = \frac{\Delta_{x_2}}{\Delta}; x_3 = \frac{\Delta_{x_3}}{\Delta}$$

If $\Delta \neq 0$, the system is consistent and determined – there is a unique solution.

If $\Delta = 0$, but at least one of $\Delta_{x_1}$ or $\Delta_{x_2}$ or $\Delta_{x_2}$ is not zero, the system is inconsistent – there is no solution.

If $\Delta = 0$ and $\Delta_{x_1} = \Delta_{x_2} = \Delta_{x_3}$, the system is consistent but undetermined – there is an infinite number of solutions**.**

*Example 1*

Choose one of the answers: 1) compatible and indeterminate; 2) compatible and determinate; 3) incompatible:

$$\begin{cases} x + 2y + z = 5 \\ 2x + 2y + z = 6. \\ x + 2y + 3z = 9 \end{cases}$$

*Solution*

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

$$\det A = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{vmatrix} = (-1)^{1+1} \cdot 1 \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + (-1)^{1+2} \cdot 2 \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + (-1)^{1+3} \cdot 1 \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix}$$
$$= (6-2) - 2(6-1) + (4-2) = 4 - 10 + 2 = -4$$

*Answer:* Compatible and determinate, $\det A = -4 \neq 0$

*Example 2*

Choose one of the answers: 1) compatible and indeterminate; 2) compatible and determinate; 3) incompatible.

$$\begin{cases} 4x + 2y - 3z = 6 \\ x + 2y + 3z = 3 \\ 2x - 2y - 9z = 2 \end{cases}$$

*Solution*

$$A = \begin{pmatrix} 4 & 2 & -3 \\ 1 & 2 & 3 \\ 2 & -2 & -9 \end{pmatrix}$$

$$\det A = \begin{vmatrix} 4 & 2 & -3 \\ 1 & 2 & 3 \\ 2 & -2 & -9 \end{vmatrix} = (-1)^3 \begin{vmatrix} 2 & -3 \\ -2 & -9 \end{vmatrix} + (-1)^4 \cdot 2 \begin{vmatrix} 4 & -3 \\ 2 & -9 \end{vmatrix} + (-1)^5 \cdot 3 \begin{vmatrix} 4 & 2 \\ 2 & -2 \end{vmatrix}$$
$$= -(-18-6) + 2\big(-36-(-6)\big) - 3(-8-4) = 24 - 60 + 36 = 0$$

$$\Delta_x = \begin{vmatrix} 6 & 2 & -3 \\ 3 & 2 & 3 \\ 2 & -2 & -9 \end{vmatrix} = 24$$

*Answer*: Incompatible

*Example 3*

Solve the system using Cramer's rule.

$$\begin{cases} 2 \cdot x_1 + 3 \cdot x_2 + 2 \cdot x_3 = 1 \\ 3 \cdot x_1 + 2 \cdot x_2 + 3 \cdot x_3 = -1 \\ 2 \cdot x_1 + 7 \cdot x_2 + 4 \cdot x_3 = -2 \end{cases}$$

*Solution*

$$A = \begin{pmatrix} 2 & 3 & 2 \\ 3 & 2 & 3 \\ 2 & 7 & 4 \end{pmatrix}$$

Let us write down Cramer's formulas:

$x_1 = \frac{\Delta_{x_1}}{\Delta}; x_2 = \frac{\Delta_{x_2}}{\Delta}; x_3 = \frac{\Delta_{x_3}}{\Delta}.$

We will find the determinant $\Delta$ of the system and the determinants of the unknowns $\Delta_{x_1}$, $\Delta_{x_2}$, and $\Delta_{x_3}$.

$$\Delta = \begin{vmatrix} 2 & 3 & 2 \\ 3 & 2 & 3 \\ 2 & 7 & 4 \end{vmatrix} = (-1)^2 \cdot 2 \cdot \begin{vmatrix} 2 & 3 \\ 7 & 4 \end{vmatrix} + (-1)^3 \cdot 3 \cdot \begin{vmatrix} 3 & 3 \\ 2 & 4 \end{vmatrix} + (-1)^4 \cdot 2 \cdot \begin{vmatrix} 3 & 2 \\ 2 & 7 \end{vmatrix}$$
$$= -26 - 18 + 34 = -10 \neq 0$$

$$\Delta_{x_1} = \begin{vmatrix} 1 & 3 & 2 \\ -1 & 2 & 3 \\ -2 & 7 & 4 \end{vmatrix} = (-1)^2 \cdot 1 \cdot \begin{vmatrix} 2 & 3 \\ 7 & 4 \end{vmatrix} + (-1)^3 \cdot 3 \cdot \begin{vmatrix} -1 & 3 \\ -2 & 4 \end{vmatrix} + (-1)^4 \cdot 2 \cdot \begin{vmatrix} -1 & 2 \\ -2 & 7 \end{vmatrix}$$
$$= -13 - 6 - 6 = -25$$

$$\Delta_{x_2} = \begin{vmatrix} 2 & 1 & 2 \\ 3 & -1 & 3 \\ 2 & -2 & 4 \end{vmatrix} = (-1)^2 \cdot 2 \cdot \begin{vmatrix} -1 & 3 \\ -2 & 4 \end{vmatrix} + (-1)^3 \cdot 1 \cdot \begin{vmatrix} 3 & 3 \\ 2 & 4 \end{vmatrix} + (-1)^4 \cdot 2 \cdot \begin{vmatrix} 3 & -1 \\ 2 & -2 \end{vmatrix}$$
$$= 4 - 6 - 8 = -10$$

$$\Delta_{x_3} = \begin{vmatrix} 2 & 3 & 1 \\ 3 & 2 & -1 \\ 2 & 7 & -2 \end{vmatrix} = (-1)^2 \cdot 2 \cdot \begin{vmatrix} 2 & -1 \\ 7 & -2 \end{vmatrix} + (-1)^3 \cdot 3 \cdot \begin{vmatrix} 3 & -1 \\ 2 & -2 \end{vmatrix} + (-1)^4 \cdot 1 \cdot \begin{vmatrix} 3 & 2 \\ 2 & 7 \end{vmatrix}$$
$$= 6 + 12 + 17 = 35$$

$$x_1 = \frac{-25}{-10}; x_2 = \frac{-10}{-10}; x_3 = \frac{35}{-10}$$

$$x_1 = 2.5; x_2 = 1; x_3 = -3.5$$

*Example 4*

Solve the system using Cramer's rule.

$$\begin{cases} 3x_1 + 5x_2 + 5x_3 + x_4 = 2 \\ 3x_1 + x_2 + 2x_3 + 2x_4 = 0 \\ x_1 + x_2 + 2x_3 + 2x_4 = 3 \end{cases}$$

$$x_4 = -1, x_3 = ?$$

*Solution*

$$\begin{cases} 3x_1 + 5x_2 + 5x_3 + (-1) = 2 \\ 3x_1 + x_2 + 2x_3 + 2 \cdot (-1) = 0 \\ x_1 + x_2 + 2x_3 + 2 \cdot (-1) = 3 \end{cases}$$

$$\begin{cases} 3x_1 + 5x_2 + 5x_3 - 1 = 2 \\ 3x_1 + x_2 + 2x_3 - 2 = 0 \\ x_1 + x_2 + 2x_3 - 2 = 3 \end{cases}$$

$$\begin{cases} 3x_1 + 5x_2 + 5x_3 = 3 \\ 3x_1 + x_2 + 2x_3 = 2 \\ x_1 + x_2 + 2x_3 = 5 \end{cases}$$

$$x_3 = \frac{\Delta_{x_3}}{\Delta}$$

$$A = \begin{pmatrix} 3 & 5 & 5 \\ 3 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 3 & 5 & 5 \\ 3 & 1 & 2 \\ 1 & 1 & 2 \end{vmatrix} = -10$$

$$\Delta_{x_3} = \begin{vmatrix} 3 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 1 & 5 \end{vmatrix} = -50$$

$$x_3 = \frac{\Delta_{x_3}}{\Delta} = \frac{-50}{-10} = 5$$

## 2.2. Solution of general systems of linear equations

Let a system of $m$ linear equations with $n$ unknowns be given:

$$\begin{cases} a_{11} \cdot x_1 + a_{12} \cdot x_2 + \ldots + a_{1n} \cdot x_n = b_1 \\ a_{21} \cdot x_1 + a_{22} \cdot x_2 + \ldots + a_{2n} \cdot x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1} \cdot x_1 + a_{m2} \cdot x_2 + \ldots + a_{mn} \cdot x_n = b_m \end{cases}.$$

*The Kronecker–Capelli theorem*

A system of linear algebraic equations is consistent (i.e., has at least one solution) if and only if the rank of the coefficient matrix is equal to the rank of the augmented matrix, that is:

$$r(A) = r(\bar{A}).$$

Let $r(A) = r(\bar{A}) = r$ the system be consistent.

*Statement 1.* If $r = n$ (where $n$ is the number of unknowns), then the system has a unique solution – the system is determined.

*Statement 2.* If $r < n$, then the system has infinitely many solutions – the system is undetermined (or indeterminate).

*Rules for solving an arbitrary system of linear equations*

1. Find the ranks of the coefficient matrix $A$ and the augmented matrix $\bar{A}$.
   If $r(A) \neq r(\bar{A})$ , then the system is inconsistent (has no solutions).
2. If $r(A) = r(\bar{A})$, then the system is consistent. Find a basic (non-zero) minor of order $r$. Select $r$ equations whose coefficients form this basic minor (discard the other equations). Keep the leading (dependent) variables on the left-hand side, and move the free variables to the right-hand side of the equations.
3. Solve the system with respect to the leading variables by expressing them in terms of the free variables. This yields the general solution of the system.
4. By assigning arbitrary values to the free variables, compute the corresponding values of the leading variables – that is, find a particular solution of the system.

Leading (basic) variables are the unknowns whose coefficients are included in the basic minor.

Free variables are the $n - r$ unknowns whose coefficients are not part of the basic minor.

*Example*

Solve the system.

$$\begin{cases} x_1 + 5x_2 + 4x_3 + 3x_4 = 1 \\ 2x_1 - x_2 + 2x_3 - x_4 = 0 \\ 5x_1 + 3x_2 + 8x_3 + x_4 = 1 \end{cases}$$

*Solution*

1. Let us find the ranks of the coefficient matrix and the augmented matrix of the system.

$$A = \begin{pmatrix} 1 & 5 & 4 & 3 \\ 2 & -1 & 2 & -1 \\ 5 & 3 & 8 & 1 \end{pmatrix} \begin{matrix} 1.r \cdot \overset{\frown}{(-2)} + 2.r \\ 1.r \cdot (-5) + 3.r \end{matrix} \begin{pmatrix} 1 & 5 & 4 & 3 \\ 0 & -11 & -6 & -7 \\ 0 & -22 & -12 & -14 \end{pmatrix}$$

$$2.\ r \cdot \overset{\frown}{(-2)} + 3.r \begin{pmatrix} 1 & 5 & 4 & 3 \\ 0 & -11 & -6 & -7 \\ 0 & -22 & -12 & -14 \end{pmatrix} \sim \begin{pmatrix} 1 & 5 & 4 & 3 \\ 0 & -11 & -6 & -7 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 5 & 4 & 3 \\ 0 & -11 & -6 & -7 \end{pmatrix}$$

$$M_2 = \begin{vmatrix} 1 & 5 \\ 0 & -11 \end{vmatrix} = -11 \neq 0$$

$$r(A) = 2$$

$$\bar{A} = \begin{pmatrix} 1 & 5 & 4 & 3 & 1 \\ 2 & -1 & 2 & -1 & 0 \\ 5 & 3 & 8 & 1 & 1 \end{pmatrix} \begin{matrix} 1.r \cdot \overset{\frown}{(-2)} + 2.r \\ 1.r \cdot (-5) + 3.r \end{matrix} \begin{pmatrix} 1 & 5 & 4 & 3 & 1 \\ 0 & -11 & -6 & -7 & -2 \\ 0 & -22 & -12 & -14 & -4 \end{pmatrix}$$

$$2.r\cdot\widetilde{(-2)}+3.r\begin{pmatrix}1 & 5 & 4 & 3 & 1\\ 0 & -11 & -6 & -7 & -2\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}\sim\begin{pmatrix}1 & 5 & 4 & 3 & 1\\ 0 & -11 & -6 & -7 & -2\end{pmatrix}$$

$$M_2=\begin{vmatrix}1 & 5\\ 0 & -11\end{vmatrix}=-11\neq 0$$

$$r(\bar{A})=2$$

2. Since $r(A)=r(\bar{A})=2$, the system is consistent.
   Let us choose a basic minor:

$$M_2=\begin{vmatrix}1 & 5\\ 2 & -1\end{vmatrix}=-11.$$

Let us take the first two equations, whose coefficients form the basic minor:

$$\begin{cases}x_1+5x_2+4x_3+3x_4=1\\ 2x_1-x_2+2x_3-x_4=0\end{cases},$$

where $x_1$and $x_2$ are the leading (basic) variables, since their coefficients are part of the basic minor; $x_3$are $x_4$ are the free variables, since the coefficients of these variables are not included in the basic minor. We keep the leading variables on the left-hand side and move the free variables to the right-hand side of the equations:

$$\begin{cases}x_1+5x_2=-4x_3-3x_4+1\\ 2x_1-x_2=-2x_3+x_4\end{cases}.$$

3. Let us express the leading variables in terms of the free variables.
   We solve the resulting system for $x_1$ and $x_2$ using Cramer's Rule**:**

$$x_1=\frac{\Delta_{x_1}}{\Delta};x_2=\frac{\Delta_{x_2}}{\Delta}$$

$$\Delta=\begin{vmatrix}1 & 5\\ 2 & -1\end{vmatrix}=-11$$

$$\Delta_{x_1}=\begin{vmatrix}-4x_3-3x_4+1 & 5\\ -2x_3+x_4 & -1\end{vmatrix}=4x_3+3x_4-1-5(-2x_3+x_4)$$
$$=4x_3+3x_4-1+10x_3-5x_4=14x_3-2x_4-1$$

$$\Delta_{x_2}=\begin{vmatrix}1 & -4x_3-3x_4+1\\ 2 & -2x_3+x_4\end{vmatrix}=-2x_3+x_4-2(-4x_3-3x_4+1)$$
$$=-2x_3+x_4+8x_3+6x_4-2=6x_3+7x_4-2$$

$$x_1=\frac{\Delta_{x_1}}{\Delta}=\frac{14x_3-2x_4-1}{-11}=-\frac{14}{11}x_3+\frac{2}{11}x_4+\frac{1}{11}$$

$$x_2=\frac{\Delta_{x_2}}{\Delta}=\frac{6x_3+7x_4-2}{-11}=-\frac{6}{11}x_3-\frac{7}{11}x_4+\frac{2}{11}$$

$$\begin{cases} x_1 = -\frac{14}{11}x_3 + \frac{2}{11}x_4 + \frac{1}{11} \\ x_2 = -\frac{6}{11}x_3 - \frac{7}{11}x_4 + \frac{2}{11} \end{cases}.$$

The general solution of the system has the following form:

$$\begin{cases} x_1 = -\frac{14}{11}c_1 + \frac{2}{11}c_2 + \frac{1}{11} \\ x_2 = -\frac{6}{11}c_1 - \frac{7}{11}c_2 + \frac{2}{11} \\ \quad x_3 = c_1,\ c_1 \in \mathbb{R} \\ \quad x_4 = c_2,\ c_2 \in \mathbb{R} \end{cases}.$$

Let us find a particular solution of the system.
Let $c_1 = 0$ and

; $c_2 = 1$, then the particular solution of the system has the following form:

$$\begin{cases} x_1 = \frac{3}{11} \\ x_2 = -\frac{5}{11} \\ x_3 = 0 \\ x_4 = 1 \end{cases}.$$

## 2.3. Solving systems of linear equations using the Gaussian elimination method

The Gaussian elimination method consists of the successive elimination of unknowns.

Let a system of linear equations be given:

$$\begin{cases} a_{11} \cdot x_1 + a_{12} \cdot x_2 + \ldots + a_{1n} \cdot x_n = b_1 \\ a_{21} \cdot x_1 + a_{22} \cdot x_2 + \ldots + a_{2n} \cdot x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1} \cdot x_1 + a_{m2} \cdot x_2 + \ldots + a_{mn} \cdot x_n = b_m \end{cases}. \qquad (1)$$

The solution process consists of two main stages.

Using elementary row operations, the system is transformed into a row echelon form (in particular, a triangular form):

$$\begin{cases} a_{11} \cdot x_1 + a_{12} \cdot x_2 + \ldots + a_{1k} \cdot x_k + \cdots + a_{1n} \cdot x_n = b_1 \\ \qquad a_{22} \cdot x_2 + \ldots + a_{2k} \cdot x_k + \cdots + a_{2n} \cdot x_n = b_2 \\ \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \qquad\qquad a_{kk} \cdot x_k + \ldots + a_{kn} \cdot x_n = b_k \end{cases}, \qquad (2)$$

where $k \le n, a_{ii} \ne 0, i = \overline{1,k}$.

The coefficients $a_{ii}$ are called the pivots or leading elements of the system.

Let us now describe how to transform System (1) into Form (2).

Assume that $a_{ii} \neq 0$. If $a_{ii} = 0$, we replace the first equation in the system with one that has a nonzero coefficient for $x_1$. In other words, we search for an equation in which the coefficient of $x_1$ is nozero and move it to the top of the system (i.e., we swap rows).

Using elementary row operations, we begin transforming System (1). We eliminate the variable $x_1$ from all equations except the first one. To do this, we multiply both sides of the first equation by $-\frac{a_{21}}{a_{11}}$ and add the result to the second equation of the system, term by term. Then, we multiply both sides of the first equation by $-\frac{a_{31}}{a_{11}}$ and add the result to the third equation of the system. Continuing this process, we obtain an equivalent system

$$\begin{cases} a_{11} \cdot x_1 + a_{12} \cdot x_2 + \ldots + a_{1n} \cdot x_n = b_1 \\ \quad a^{(1)}{}_{22} \cdot x_2 + \ldots + a^{(1)}{}_{2n} \cdot x_n = b^{(1)}{}_2 \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\ \quad a^{(1)}{}_{m2} \cdot x_2 + \ldots + a^{(1)}{}_{mn} \cdot x_n = b^{(1)}{}_m \end{cases},$$

where $a^{(1)}{}_{ij}$ and $b^{(1)}{}_i$ (for $i = \overline{2,m}$; $j = \overline{2,n}$) are the new values of the coefficients and the right-hand sides, obtained after the first step.

Next, in a similar manner, taking $a^{(1)}{}_{22} \neq 0$ as the pivot element, we eliminate the variable $x_2$ from all equations of the system except the first and the second, and so on.

If, in the process of transforming System (1) to row echelon form, zero equations appear, i.e., identities of the form $0 = 0$, they are discarded. However, if equations of the form $0 = b_i$ appear, where $b_i \neq 0$, then the system is inconsistent.

At the second stage, we solve the row echelon system. In the last equation of this system, we express the first unknown, $x_k$, in terms of the remaining variables $(x_{k+1}, \ldots, x_n)$. Then we substitute the obtained value of $x_k$ into the second-to-last equation and express $x_{k-1}$ $(x_{k+1}, \ldots, x_n)$ and so on, successively finding $x_{k-2}, \ldots, x_1$. By assigning arbitrary values to the free variables $(x_{k+1}, \ldots, x_n)$, we obtain the general solution set of the system.

*Remarks*

1. If the row echelon system turns out to be triangular, i.e, $k = n$, then the original system has a unique solution.
2. In practice, it is more convenient to work not with System (1) itself, but with its augmented matrix, performing all elementary row operations on its rows.

*Example 1*

Solve the linear system:

$$\begin{cases} 2x + 3y - z = 1 \\ x + 2z - y = 3 \\ 4x + y - 3z = -11 \end{cases}.$$

*Solution*

Make the leading coefficient in the first row equal to **1**. Swap Row 1 and Row 2 (since Row 2 starts with **1**):

$$\left(\begin{array}{ccc|c}2&3&-1&1\\1&-1&2&3\\4&1&-3&-11\end{array}\right)\sim\left(\begin{array}{ccc|c}1&-1&2&3\\2&3&-1&1\\4&1&-3&-11\end{array}\right)\begin{array}{l}1.r\cdot\widetilde{(-2)}+2.r\\1.r\cdot(-4)+3.r\end{array}\left(\begin{array}{ccc|c}1&-1&2&3\\0&5&-5&-5\\0&5&-11&-23\end{array}\right)2.r\cdot\widetilde{(-1)}+3.r$$

$$\left(\begin{array}{ccc|c}1&-1&2&3\\0&5&-5&-5\\0&0&-6&-18\end{array}\right).$$

Write the new system:

$$\begin{cases}x-y+2z=3\\5y-5z=-5\\-6z=-18\end{cases}$$

*Back substitution*

From Row 3:

$$-6z=-18$$

$$z=3$$

From Row 2:

$$5y-5\cdot 3=-5$$

$$y=2$$

From Row 1:

$$x-2+2\cdot 3=3$$

$$x=-1$$

The answer: $(-1;2:3)$

*Example 2*

Solve the linear system:

$$\begin{cases}2x+3y+z=3\\x-4y+z=-12\\3x+2z-y=-9\end{cases}$$

*Solution*

Make the leading coefficient in the first row equal to **1**. Swap Row 1 and Row 2 (since Row 2 starts with **1**):

$$\left(\begin{array}{ccc|c}2&3&1&3\\1&-4&1&-12\\3&-1&2&-9\end{array}\right)\sim\left(\begin{array}{ccc|c}1&-4&1&-12\\2&3&1&3\\3&-1&2&-9\end{array}\right)\begin{array}{l}1.r\cdot\widetilde{(-2)}+2.r\\1.r\cdot(-3)+3.r\end{array}\left(\begin{array}{ccc|c}1&-4&1&-12\\0&11&-1&27\\0&11&-1&27\end{array}\right)2.r\cdot\widetilde{(-1)}+3.r$$

$$\left(\begin{array}{ccc|c} 1 & -4 & 1 & -12 \\ 0 & 11 & -1 & 27 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Write the new system:

$$\begin{cases} x - 4y + z = -12 \\ 11y - z = 27 \\ 0z = 0 \end{cases}$$

From Row 3:

$$0z = 0$$

$$z \in \mathbb{R}$$

From Row 2:

$$11y - z = 27$$

$$y = \frac{27 + z}{11}$$

From Row 1:

$$x - 4 \cdot \left(\frac{27 + z}{11}\right) + z = -12$$

$$x = -12 + \frac{108 + 4z - 11z}{11}$$

$$x = -12 + \frac{108 - 7z}{11}$$

*The answer:* $x = -12 + \frac{108-7z}{11}; y = \frac{27+z}{11}; z \in \mathbb{R}$

*Example 3*

Solve the linear system:

$$\begin{cases} x + y + z = 3 \\ 2x + 2z + 2y = 6 \\ x + y + z = 4 \end{cases}$$

*Solution*

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & 6 \\ 1 & 1 & 1 & 4 \end{array}\right) \begin{array}{c} 1.r \cdot \widetilde{(-2)} + 2.r \\ 1.r \cdot (-1) + 3.r \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

Write the new system:

$$\begin{cases} x + y + 1z = 3 \\ 0z = 1 \end{cases}$$

*Back substitution*

From Row 3:

$$0z = 1$$

$$z \in \emptyset$$

*The answer:* The system is inconsistent and has no solution.

**Applications**

*Problem 1*

Data on the production of three types of agricultural products – grain, milk, and meat (in conventional units) – in two farms for the years 2023 and 2024 are given in the form of matrices:

$$A_{2023} = \begin{pmatrix} 2053 & 571 & 340 \\ 2679 & 453 & 694 \end{pmatrix}, A_{2024} = \begin{pmatrix} 2745 & 644 & 800 \\ 2900 & 451 & 373 \end{pmatrix}.$$

Find:

a) the total volume of products produced over the two years (2023 and 2024);
b) the growth in production volumes in 2023 compared to 2024, by product types and by farming enterprises;
c) the matrix of average annual production of products.

You should also explain the economic meaning of the elements in the resulting matrices.

*Solution*

Rows: Row 1 – Farm 1; Row 2 –Farm 2. Columns: Column 1 – grain; Column 2 – milk; Column 3 – meat.

a) Total volume of products produced over the two years (2023 and 2024):

$$A_{total} = A_{2023} + A_{2024} = \begin{pmatrix} 2053 & 571 & 340 \\ 2679 & 453 & 694 \end{pmatrix} + \begin{pmatrix} 2745 & 644 & 800 \\ 2900 & 451 & 373 \end{pmatrix}$$
$$= \begin{pmatrix} 4798 & 1215 & 1140 \\ 5579 & 904 & 1067 \end{pmatrix}$$

Economic meaning of the resulting matrix:

Total production matrix shows the combined volume of grain, milk, and meat produced by each farm over two years. For example, Farm 1 produced 4798 units of grain in total.

b) Growth in production volumes in 2024 compared to 2023:

$$A_{growth} = A_{2024} - A_{2023} = \begin{pmatrix} 2745 & 644 & 800 \\ 2900 & 451 & 373 \end{pmatrix} - \begin{pmatrix} 2053 & 571 & 340 \\ 2679 & 453 & 694 \end{pmatrix}$$
$$= \begin{pmatrix} 692 & 73 & 460 \\ 221 & -2 & -321 \end{pmatrix}$$

Economic meaning of the resulting matrix:

Growth matrix shows how each product's output changed from 2023 to 2024. For example, Farm 1 increased meat production by 460 units**,** while Farm 2 decreased meat production by 321 units**.**

*Problem 2*

Problem

Data on the production of three types of home appliances – refrigerators**,** washing machines, and microwaves (in units) – at two factories in the years 2023 and 2024 are given in the form of matrices:

$$A_{2023} = \begin{pmatrix} 1200 & 850 & 640 \\ 1350 & 920 & 780 \end{pmatrix}, A_{2024} = \begin{pmatrix} 1250 & 870 & 690 \\ 1450 & 950 & 760 \end{pmatrix}.$$

Find: The matrix of average annual production of products.

*Solution*

Rows: Row 1 – Factory 1; Row 2 – Factory 2. Columns: Column 1 – refrigerators; Column 2 – washing machines; Column 3 – microwaves.

To find the average annual production, we calculate:

$$A_{avg} = \frac{1}{2}(A_{2023} + A_{2024}).$$

First, add the matrices:

$$A_{total} = A_{2023} + A_{2024} = \begin{pmatrix} 1200 & 850 & 640 \\ 1350 & 920 & 780 \end{pmatrix} + \begin{pmatrix} 1250 & 870 & 690 \\ 1450 & 950 & 760 \end{pmatrix}$$
$$= \begin{pmatrix} 2450 & 1720 & 1330 \\ 2800 & 1870 & 1540 \end{pmatrix}$$

$$A_{avg} = \frac{1}{2}(A_{2023} + A_{2024}) = \frac{1}{2}\begin{pmatrix} 2450 & 1720 & 1330 \\ 2800 & 1870 & 1540 \end{pmatrix} = \begin{pmatrix} 1225 & 860 & 665 \\ 1400 & 935 & 770 \end{pmatrix}.$$

*Economic meaning of the resulting matrix:*

The average matrix provides an estimate of annual production, which helps in forecasting future trends and assessing factory performance.
For example, Factory 2 produces, on average, 1400 refrigerators**,** 935 washing machines, and 770 microwaves per year.

*Problem 3*

Two companies manufacture desk lamps**,** office chairs, and bookshelves. The quantity of each product manufactured per month is shown in Table 1.

Table 1

| Company | Desk lamps | Office chairs | Bookshelves |
|---|---|---|---|
| Company 1 | 90 | 150 | 75 |
| Company 2 | 120 | 100 | 90 |

The unit prices of each product in euros for three consecutive months are given in Table 2.

Table 2

| Product | March | April | May |
|---|---|---|---|
| Desk lamps | 35 | 36 | 38 |
| Office chairs | 58 | 60 | 59 |
| Bookshelves | 72 | 75 | 78 |

Tasks

a) Construct a matrix of product quantities for each company.
b) Construct a matrix of unit prices for the three months.
c) Compute the revenue matrix for each company for each month by multiplying the quantity matrix by the price matrix.
d) Explain the economic meaning of the resulting matrix elements.

*Solution*

Let $A$, the quantity matrix, rows represent companies and columns represent products (lamps, chairs, shelves):

$$A = \begin{pmatrix} 90 & 150 & 75 \\ 120 & 100 & 90 \end{pmatrix}.$$

Let $B$, the unit price matrix, rows represent products and columns represent months:

$$B = \begin{pmatrix} 35 & 36 & 38 \\ 58 & 60 & 59 \\ 72 & 75 & 78 \end{pmatrix}.$$

Revenue matrix $R = A \cdot B$, where each element $r_{ij}$ is the total revenue of company $i$ in month $j$.

$$R = \begin{pmatrix} 90 & 150 & 75 \\ 120 & 100 & 90 \end{pmatrix} \begin{pmatrix} 35 & 36 & 38 \\ 58 & 60 & 59 \\ 72 & 75 & 78 \end{pmatrix}$$
$$= \begin{pmatrix} 90 \cdot 35 + 150 \cdot 58 + 75 \cdot 72 & 90 \cdot 36 + 150 \cdot 60 + 75 \cdot 75 & 90 \cdot 38 + 150 \cdot 59 + 75 \cdot 78 \\ 120 \cdot 35 + 100 \cdot 58 + 90 \cdot 72 & 120 \cdot 36 + 100 \cdot 60 + 90 \cdot 75 & 120 \cdot 38 + 100 \cdot 59 + 90 \cdot 78 \end{pmatrix}$$
$$= \begin{pmatrix} 17250 & 17865 & 18120 \\ 16480 & 17070 & 17480 \end{pmatrix}$$

*Economic meaning*

Each element $r_{ij}$ is the total revenue of company $i$ in month $j$.

For example:

The first company earned 17 250 euros in March**,** 17 865 euros in April, and 18 120 euros in May.

The second company earned 16 480 euros in March, and so on.

*Problem 4*

A European manufacturer plans to produce the following number of units in September: 600 electric kettles, 950 toasters and 400 microwave ovens.

To manufacture these appliances, the company uses five types of materials: steel, plastic, glass, copper wire, and insulation. The required amount of each material (in kg per unit) and the cost per kilogram in euros are provided below.

Material Requirements Per Unit (kg)

| Appliance | Steel | Plastic | Glass | Copper wire | Insulation |
|---|---|---|---|---|---|
| Electric kettles | 1.2 | 0.8 | 0.4 | 0.2 | 0.1 |
| Toasters | 1.0 | 1.1 | 0.2 | 0.3 | 0.2 |
| Microwave ovens | 2.5 | 1.8 | 1.0 | 0.5 | 0.3 |

Cost per kilogram of materials (in euros): Steel: €3.20; plastic: €2.40; glass: €1.80; copper wire: €6.00; insulation: €2.00.

Tasks:

a) Calculate the total quantity of each material needed for the entire production plan.
b) Determine the material cost per unit of each appliance.
c) Find the total material cost for producing all planned units.

*Solution*

Let the plan vector be

$$X = (600 \quad 950 \quad 400),$$

the material requirement matrix per unit (rows: appliances, columns: materials) be

$$M = \begin{pmatrix} 1.2 & 0.8 & 0.4 & 0.2 & 0.1 \\ 1.0 & 1.1 & 0.2 & 0.3 & 0.2 \\ 2.5 & 1.8 & 1.0 & 0.5 & 0.3 \end{pmatrix},$$

and the material cost vector (in €/kg) be

$$C = \begin{pmatrix} 3.2 \\ 2.4 \\ 1.8 \\ 6 \\ 2 \end{pmatrix}.$$

We compute the total material needed as

$$Q = X \cdot M = (600 \quad 950 \quad 400) \begin{pmatrix} 1.2 & 0.8 & 0.4 & 0.2 & 0.1 \\ 1.0 & 1.1 & 0.2 & 0.3 & 0.2 \\ 2.5 & 1.8 & 1.0 & 0.5 & 0.3 \end{pmatrix}.$$
$$= (2670 \quad 2245 \quad 830 \quad 605 \quad 370)$$

Interpretation of Q: 2670 kg of steel; 2245 kg of plastic; 830 kg of glass; 605 kg of copper wire, and 370 kg of insulation.

Material cost per unit of each appliance:

$$D = M \cdot C = \begin{pmatrix} 1.2 & 0.8 & 0.4 & 0.2 & 0.1 \\ 1.0 & 1.1 & 0.2 & 0.3 & 0.2 \\ 2.5 & 1.8 & 1.0 & 0.5 & 0.3 \end{pmatrix} \begin{pmatrix} 3.2 \\ 2.4 \\ 1.8 \\ 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 7.88 \\ 8.40 \\ 17.72 \end{pmatrix}.$$

The material cost per unit is 7.88 euros for an electric kettle, 8.40 euros for a toaster, and 17.72 euros for a microwave oven.

Total cost for the entire production is

$$C_{total} = X \cdot D = (600 \quad 950 \quad 400) \begin{pmatrix} 7.88 \\ 8.40 \\ 17.72 \end{pmatrix} = 600 \cdot 7.88 + 950 \cdot 8.40 + 400 \cdot 17.72$$
$$= 19796.$$

The total material cost for producing all planned units is 19 796 euros.

*Problem 5*

A company has three workshops, each producing a different type of product: Workshop 1 produces product type 1; Workshop 2 produces product type 2; Workshop 3 produces product type 3. Part of the output is used for internal consumption, while the rest is the final product delivered outside the system.

Tasks:

1. Determine the distribution of products among workshops used for internal consumption.
2. Find the total output of each workshop, given that:

the input coefficient matrix (direct consumption matrix)

$$A = \begin{pmatrix} 0.6 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.1 & 0.7 \end{pmatrix},$$

And the final demand vector

$$Y = \begin{pmatrix} 120 \\ 90 \\ 60 \end{pmatrix}.$$

*Solution*

Let us denote the matrix of total output volumes as $X$.
Since the produced output $X$ is used both for internal consumption in the amount $A \cdot X$ (where matrix $A$ is the direct consumption matrix), and as the final product $Y$, the following matrix equation holds:

$$X - A \cdot X = Y \text{ or } (E - A) \cdot X = Y,$$

where $E$ is the identity matrix. The solution to this equation is found using the formula

$$X = (E - A)^{-1} \cdot Y$$

$$E - A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.6 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.1 & 0.7 \end{pmatrix} = \begin{pmatrix} 0.4 & -0.2 & -0.1 \\ -0.3 & 0.5 & -0.2 \\ -0.2 & -0.1 & 0.3 \end{pmatrix}.$$

Let us calculate the inverse matrix $(E - A)^{-1}$.

$$\det(E - A) = \begin{vmatrix} 0.4 & -0.2 & -0.1 \\ -0.3 & 0.5 & -0.2 \\ -0.2 & -0.1 & 0.3 \end{vmatrix}$$
$$= (-1)^2 \cdot 0.4 \cdot \begin{vmatrix} 0.5 & -0.2 \\ -0.1 & 0.3 \end{vmatrix} + (-1)^3 \cdot (-0.2) \cdot \begin{vmatrix} -0.3 & -0.2 \\ -0.2 & 0.3 \end{vmatrix} + (-1)^4$$
$$\cdot (-0.1) \cdot \begin{vmatrix} -0.3 & 0.5 \\ -0.2 & -0.1 \end{vmatrix} = 0.4 \cdot 0.13 + 0.2 \cdot (-0.13) - 0.1 \cdot 0.13 = 0.013 \neq 0$$

Since $|E - A| \neq 0$, the matrix $E - A$ is non-singular. Let us construct the adjugate matrix:

$$(E - A)^* = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

| | | |
|---|---|---|
| $A_{11} = (-1)^{1+1} \begin{vmatrix} 0.5 & -0.2 \\ -0.1 & 0.3 \end{vmatrix} = 0.13$ | $A_{21} = (-1)^{2+1} \begin{vmatrix} -0.2 & -0.1 \\ -0.1 & 0.3 \end{vmatrix} = 0.07$ | $A_{31} = (-1)^{3+1} \begin{vmatrix} -0.2 & -0.1 \\ 0.5 & -0.2 \end{vmatrix} = 0.09$ |

| | | |
|---|---|---|
| $A_{12} = (-1)^{1+2}\begin{vmatrix}-0.3 & -0.2\\ -0.2 & 0.3\end{vmatrix} = 0.13$ | $A_{22} = (-1)^{2+2}\begin{vmatrix}0.4 & -0.1\\ -0.2 & 0.3\end{vmatrix} = 0.10$ | $A_{32} = (-1)^{3+2}\begin{vmatrix}0.4 & -0.1\\ -0.3 & -0.2\end{vmatrix} = 0.11$ |
| $A_{13} = (-1)^{1+3}\begin{vmatrix}-0.3 & 0.5\\ -0.2 & -0.1\end{vmatrix} = 0.13$ | $A_{23} = (-1)^{2+3}\begin{vmatrix}0.4 & -0.2\\ -0.2 & -0.1\end{vmatrix} = 0.08$ | $A_{33} = (-1)^{3+3}\begin{vmatrix}0.4 & -0.2\\ -0.3 & 0.5\end{vmatrix} = 0.14$ |

$$(E-A)^* = \begin{pmatrix}0.13 & 0.13 & 0.13\\ 0.07 & 0.10 & 0.08\\ 0.09 & 0.11 & 0.14\end{pmatrix}.$$

Transpose the cofactor matrix (adjugate):

$$((E-A)^*)^T = \begin{pmatrix}0.13 & 0.07 & 0.09\\ 0.13 & 0.10 & 0.11\\ 0.13 & 0.11 & 0.14\end{pmatrix}$$

$$(E-A)^{-1} = \frac{1}{0.013}\begin{pmatrix}0.13 & 0.07 & 0.09\\ 0.13 & 0.10 & 0.11\\ 0.13 & 0.11 & 0.14\end{pmatrix} = \begin{pmatrix}10 & 5.38 & 6.92\\ 10 & 7.69 & 8.46\\ 10 & 6.15 & 10.77\end{pmatrix}$$

$$X = (E-A)^{-1}\cdot Y = \begin{pmatrix}10 & 5.38 & 6.92\\ 10 & 7.69 & 8.46\\ 10 & 6.15 & 10.77\end{pmatrix}\begin{pmatrix}120\\ 90\\ 60\end{pmatrix} = \begin{pmatrix}2099.40\\ 2399.70\\ 2399.70\end{pmatrix}$$

Total output of Workshop 1**:** 2099.40 units; total output of Workshop 2**:** 2399.70 units; and total output of Workshop 2**:** 2399.70 units. These values represent the total production required from each workshop to both meet the external demand and satisfy internal consumption needs according to the specified input-output relationships.

The distribution of products among workshops for internal consumption is determined by the formulas:

$$x_{ij} = a_{ij}\cdot x_j$$

| | | |
|---|---|---|
| $x_{11} = 0.6\cdot 2099.40 = 1259.64$ | $x_{21} = 0.3\cdot 2099.40 = 629.82$ | $x_{31} = 0.2\cdot 2099.40 = 419.88$ |
| $x_{12} = 0.2\cdot 2399.70 = 479.94$ | $x_{22} = 0.5\cdot 2399.70 = 1199.85$ | $x_{32} = 0.1\cdot 2399.70 = 239.97$ |
| $x_{13} = 0.1\cdot 2399.70 = 239.97$ | $x_{23} = 0.2\cdot 2399.70 = 479.94$ | $x_{33} = 0.7\cdot 2399.70 = 1679.79$ |

*Internal consumption and total output*

| Internal consumption | Workshop 1 | Workshop 2 | Workshop 3 | Final product (Y) | Total output(X) |
|---|---|---|---|---|---|
| Workshop 1 | 1259.64 | 479.94 | 239.97 | 120 | 2099.40 |
| Workshop 2 | 629.82 | 1199.85 | 479.94 | 90 | 2399.70 |
| Workshop 3 | 419.88 | 239.97 | 1679.79 | 60 | 2399.70 |
| Net product | 420.06 | 479.94 | 240.00 | — | — |
| Total output | 2099.40 | 2399.70 | 2399.70 | — | 6898.80 |

*Problem 6*

A manufacturing plant purchased 50 kg of Steel Type A, 60 kg of Steel Type B, and 40 kg of Steel Type C for producing machine parts. The total cost of the purchase was 21 000 monetary units. Find the price per kilogram of each steel type if: 40 kg of Steel A costs the same as 60 kg of Steel B; 20 kg of Steel C is 300 monetary units cheaper than 30 kg of Steel A.

*Solution 1*

Let $x_1, x_2,$ and $x_3$ be the cost (in monetary units) per kilogram of Steel A, B, and C, respectively. From the conditions we get the following system of equations:

$$\begin{cases} 50x_1 + 60x_2 + 40x_3 = 21000 \\ 40x_1 = 60x_2 \\ 30x_1 - 20x_3 = 300 \end{cases},$$

or equivalently:

$$\begin{cases} 50x_1 + 60x_2 + 40x_3 = 21000 \\ 40x_1 - 60x_2 = 0 \\ 30x_1 - 20x_3 = 300 \end{cases}.$$

We compute the determinant of the coefficient matrix

$$\det A = \begin{vmatrix} 50 & 60 & 40 \\ 40 & -60 & 0 \\ 30 & 0 & -20 \end{vmatrix} = (-1)^2 \cdot 50 \cdot \begin{vmatrix} -60 & 0 \\ 0 & -20 \end{vmatrix} + (-1)^3 \cdot 60 \cdot \begin{vmatrix} 40 & 0 \\ 30 & -20 \end{vmatrix} + (-1)^4 \cdot 40 \cdot \begin{vmatrix} 40 & -60 \\ 30 & 0 \end{vmatrix} = 60000 + 48000 + 72000 = 180000 \neq 0.$$

Since the determinant is nonzero, the system has a unique solution. Let us find the solution using the inverse matrix, i.e., by the formula $X = A^{-1} \cdot B$.

The coefficient matrix of the system

$$A = \begin{pmatrix} 50 & 60 & 40 \\ 40 & -60 & 0 \\ 30 & 0 & -20 \end{pmatrix}.$$

The column of constants (right-hand side)

$$B = \begin{pmatrix} 21000 \\ 0 \\ 300 \end{pmatrix}.$$

$$(A)^* = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

| | | |
|---|---|---|
| $A_{11} = (-1)^{1+1} \begin{vmatrix} -60 & 0 \\ 0 & -20 \end{vmatrix} = 1200$ | $A_{21} = (-1)^{2+1} \begin{vmatrix} 60 & 40 \\ 0 & -20 \end{vmatrix} = 1200$ | $A_{31} = (-1)^{3+1} \begin{vmatrix} 60 & 40 \\ -60 & 0 \end{vmatrix} = 2400$ |
| $A_{12} = (-1)^{1+2} \begin{vmatrix} 40 & 0 \\ 30 & -20 \end{vmatrix} = 800$ | $A_{22} = (-1)^{2+2} \begin{vmatrix} 50 & 40 \\ 30 & -20 \end{vmatrix} = -2200$ | $A_{32} = (-1)^{3+2} \begin{vmatrix} 50 & 40 \\ 40 & 0 \end{vmatrix} = 1600$ |
| $A_{13} = (-1)^{1+3} \begin{vmatrix} 40 & -60 \\ 30 & 0 \end{vmatrix} = 1800$ | $A_{23} = (-1)^{2+3} \begin{vmatrix} 50 & 60 \\ 30 & 0 \end{vmatrix} = -1800$ | $A_{33} = (-1)^{3+3} \begin{vmatrix} 50 & 60 \\ 40 & -60 \end{vmatrix} = -5400$ |

$$(A)^* = \begin{pmatrix} 1200 & 800 & 1800 \\ 1200 & -2200 & -1800 \\ 2400 & 1600 & -5400 \end{pmatrix}$$

Transpose the cofactor matrix (adjugate):

$$(A^*)^T = \begin{pmatrix} 1200 & 1200 & 2400 \\ 800 & -2200 & 1600 \\ 1800 & -1800 & -5400 \end{pmatrix}$$

$$\begin{aligned}(A)^{-1} &= \frac{1}{180000} \begin{pmatrix} 1200 & 1200 & 2400 \\ 800 & -2200 & 1600 \\ 1800 & -1800 & -5400 \end{pmatrix} \begin{pmatrix} 21000 \\ 0 \\ 300 \end{pmatrix} = \frac{1}{180000} \begin{pmatrix} 1200 \cdot 21000 + 2400 \cdot 300 \\ 800 \cdot 21000 + 1600 \cdot 300 \\ 1800 \cdot 21000 - 5400 \cdot 300 \end{pmatrix} \\ &= \frac{1}{180000} \begin{pmatrix} 25920000 \\ 17280000 \\ 36180000 \end{pmatrix} = \begin{pmatrix} 144 \\ 96 \\ 201 \end{pmatrix}\end{aligned}$$

The price per kg of Steel A is $144$, price per kg of Steel B is $96$, and price per kg of Steel C is $201$.

*Solution 2*

Let us use the Gaussian elimination method. Let us divide each equation in the system by 10.

$$\left(\begin{array}{ccc|c} 50 & 60 & 40 & 21000 \\ 40 & -60 & 0 & 0 \\ 30 & 0 & -20 & 300 \end{array}\right) \sim \left(\begin{array}{ccc|c} 5 & 6 & 4 & 2100 \\ 4 & -6 & 0 & 0 \\ 3 & 0 & -2 & 30 \end{array}\right) \begin{array}{l} \\ 1.r \cdot \left(-\frac{4}{5}\right) + 2.r \\ 1.r \cdot \left(-\frac{3}{5}\right) + 3.r \end{array} \left(\begin{array}{ccc|c} 5 & 6 & 4 & 2100 \\ 0 & -\frac{54}{5} & -\frac{16}{5} & -1680 \\ 0 & -\frac{18}{5} & -\frac{22}{5} & -1230 \end{array}\right) \sim$$

$$2.r\cdot\left(-\frac{1}{3}\right)+3.r\left(\begin{array}{ccc|c}5 & 6 & 4 & 2100\\ 0 & -\frac{54}{5} & -\frac{16}{5} & -1680\\ 0 & 0 & -\frac{10}{3} & -670\end{array}\right)$$

Write the new system:

$$\begin{cases}5x+6y+4z=2100\\ -\frac{54}{5}y-\frac{16}{5}z=-1680\\ -\frac{10}{3}z=-670\end{cases}$$

From Row 3

$$-\frac{10}{3}z=-670$$

$$z=201$$

From Row 2

$$-\frac{54}{5}y-\frac{16}{5}\cdot 201=-1680$$

$$y=96$$

From Row 1

$$5x+6\cdot 96+4\cdot 201=2100$$

$$x=144$$

The price per kg of Steel A is $144$; price per kg of Steel B is $96$; and price per kg of Steel C is $201$.

*Problem 7*

In two alloys, copper and zinc are in the ratios $5:2$ and $3:4$ (by mass), respectively. How many kilograms of the first alloy and of the second alloy should be taken so that, after remelting, we obtain $28$ kg of a new alloy with equal amounts of copper and zinc?

*Solution*

In this example, we are given not the concentration or percentage content of copper and zinc in the alloys, but their weight ratio. Let the mass of the first alloy be $x$, and the second be $y$. Since in the first alloy the mass of copper to zinc is in the ratio $5:2$ (a total of seven parts), this means that the copper content in the alloy is $\frac{5}{7}x$, and the zinc content is $\frac{2}{7}x$.

Similarly, we determine the copper and zinc content in the second alloy: copper is $\frac{3}{7}y$, and zinc is $\frac{4}{7}y$.

The total mass after remelting is $28$ kg, i.e.,

$x + y = 28.$

The amount of copper in this alloy will be $\frac{5}{7}x + \frac{3}{7}y$, and the amount of zinc will be $\frac{2}{7}x + \frac{4}{7}y$.

Since these masses are equal, we have

$$\frac{5}{7}x + \frac{3}{7}y = \frac{2}{7}x + \frac{4}{7}y,$$

which simplifies to

$$\frac{3}{7}x - \frac{1}{7}y = 0,$$

or

$$y = 3x$$

$$\begin{cases} x + y = 28 \\ y = 3x \end{cases}$$

$$x = 7\ kg$$

$y = 21\ kg.$

We need to take 7 kg of the first alloy and 21 kg of the second.

*Problem 8*

A pixel has the color values $(R, G, B) = (80, 150, 220)$.
To reduce the brightness by 15%, the following transformation matrix is applied:

$$T = \begin{pmatrix} 0.85 & 0 & 0 \\ 0 & 0.85 & 0 \\ 0 & 0 & 0.85 \end{pmatrix}.$$

What will be the new color component values of the pixel after applying this transformation?

*Solution*

$$C = \begin{pmatrix} 0.85 & 0 & 0 \\ 0 & 0.85 & 0 \\ 0 & 0 & 0.85 \end{pmatrix} \begin{pmatrix} 80 \\ 150 \\ 220 \end{pmatrix} = \begin{pmatrix} 0.85 \cdot 80 \\ 0.85 \cdot 150 \\ 0.85 \cdot 220 \end{pmatrix} = \begin{pmatrix} 68 \\ 127.50 \\ 187 \end{pmatrix}$$

A light blue color with a slight turquoise tint.

*Problem 9*

A message is encoded using the matrix

$$C = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix},$$

and the original message is represented by the vector

$$v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Find the encoded vector $Cv$.

*Solution*

Multiply the matrix by the vector

$$Cv = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 10 \end{pmatrix}.$$

Encoded vector $Cv = \begin{pmatrix} 4 \\ 10 \end{pmatrix}$.

*Problem 10*

Currents $I_1, I_2,$ and $I_3$ in a *3*-node circuit satisfy Kirchhoff's equations:

$$\begin{cases} 2I_1 - I_2 = 4 \\ -I_1 + 3I_2 - I_3 = 0. \\ -I_2 + 2I_3 = 5 \end{cases}$$

Find the currents $I_1, I_2,$ and $I_3$.

*Solution*

Let us solve the system using Gaussian elimination.

$$\left(\begin{array}{ccc|c} 2 & -1 & 0 & 4 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 2 & 5 \end{array}\right) \sim \left(\begin{array}{ccc|c} -1 & 3 & -1 & 0 \\ 2 & -1 & 0 & 4 \\ 0 & -1 & 2 & 5 \end{array}\right) 1.r \cdot \widetilde{2 +} 2.r \left(\begin{array}{ccc|c} -1 & 3 & -1 & 0 \\ 0 & 5 & -2 & 4 \\ 0 & -1 & 2 & 5 \end{array}\right) \sim$$

$$2.r \cdot \frac{1}{5} + 3.r \left(\begin{array}{ccc|c} -1 & 3 & -1 & 0 \\ 0 & 5 & -2 & 4 \\ 0 & 0 & \frac{8}{5} & \frac{29}{5} \end{array}\right)$$

Write the new system:

$$\begin{cases} -I_1 + 3I_2 - I_3 = 0 \\ 5I_2 - 2I_3 = 4 \\ \frac{8}{5} I_3 = \frac{29}{5} \end{cases}$$

From Row 3

$$\frac{8}{5}I_3 = \frac{29}{5}$$

$$I_3 = \frac{29}{8}$$

From Row 2

$$5I_2 - 2I_3 = 4$$

$$5I_2 - 2 \cdot \frac{29}{8} = 4$$

$$5I_2 = \frac{45}{4}$$

$$I_2 = \frac{9}{4}$$

From Row 1

$$-I_1 + 3I_2 - I_3 = 0$$

$$-I_1 + 3 \cdot \frac{9}{4} - \frac{29}{8} = 4$$

$$-I_1 = \frac{-54 + 29}{8}$$

$$I_1 = \frac{25}{8}$$

$I_1 = \frac{25}{8}$; $I_2 = \frac{9}{4}$;$I_3 = \frac{29}{8}$

***Exercises***

1) Let $A = \begin{pmatrix} 7 & 0 & 2 \\ 4 & 7 & 1 \\ 0 & -2 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 4 \\ -1 & -4 & 0 \\ 0 & -2 & 3 \end{pmatrix}$, then calculate $A + B$.

2) Find the difference between the matrices $C = \begin{pmatrix} -1 & 5 \\ 4 & 12 \\ 11 & -1 \end{pmatrix}$ and $G = \begin{pmatrix} 1 & 0 & -6 \\ 1 & -7 & 3 \\ -2 & 1 & 11 \end{pmatrix}$. .

3) Find the difference between $A$ and $B$. $A = \begin{pmatrix} \frac{3}{4} & \frac{7}{2} \\ \frac{1}{6} & -\frac{1}{3} \end{pmatrix}$ and $B = \begin{pmatrix} \frac{3}{2} & -\frac{5}{8} \\ \frac{1}{3} & 0 \end{pmatrix}$.

4) Add matrix $F$ and matrix $K$. $F = \begin{pmatrix} -45 & 23 \\ -12 & -63 \\ 41 & 32 \end{pmatrix}$ and $K = \begin{pmatrix} 79 & 52 \\ -36 & -11 \\ 10 & -7 \end{pmatrix}$.

5) Let $S = \begin{pmatrix} 7 & 4 & -1 \\ 2 & 0 & 4 \end{pmatrix}$ and $k = -3$, then calculate $kS$.

6) Let $= \begin{pmatrix} 6 & 2 & 1 \\ 0 & 3 & -3 \\ -4 & 8 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 & -1 \\ 5 & 5 & 0 \\ 2 & 6 & 2 \end{pmatrix}$. Find $C = 4A - \frac{1}{2}B$.

7) Compute $AB$ and $BA$, where $A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 9 & -6 \\ -3 & 2 \end{pmatrix}$.

8) Compute $AB$ and $BA$, where $A = \begin{pmatrix} 1 & 0 \\ -1 & 4 \\ 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 7 \\ -4 & 2 \\ 3 & 0 \end{pmatrix}$.

9) Compute $AB$, where $A = \begin{pmatrix} -4 & 2 & 3 & 10 & 12 \end{pmatrix}$ and $B = \begin{pmatrix} 7 \\ 2 \\ -4 \\ 2 \\ -6 \end{pmatrix}$.

10) Compute $AB$, where $A = \begin{pmatrix} 7 & 0 & 1 & -1 \\ 3 & 2 & 1 & 4 \\ 0 & 2 & -4 & 5 \\ 8 & 5 & 7 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 0 & 5 & -6 \\ 0 & 5 & 7 & 4 \\ 3 & 0 & -1 & 1 \\ 0 & 1 & 4 & 1 \end{pmatrix}$.

11) Let $A = \begin{pmatrix} 2 & 5 \\ -3 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 4 & -5 \\ 3 & k \end{pmatrix}$. What value(s) of $k$, if any, will make $AB = BA$?

12) Find $A^3$ for $A = \begin{pmatrix} 4 & -1 \\ 0 & 2 \end{pmatrix}$.

13) Given the matrix $A = \begin{pmatrix} 1 & -1 & 3 \\ 0 & 4 & 1 \\ 3 & 0 & 2 \end{pmatrix}$. Compute the matrix $B = A^2 - 3E$.

14) Let matrix $C = \begin{pmatrix} 7 & 2 & -1 \\ 4 & 7 & 2 \\ -1 & 0 & 4 \end{pmatrix}$ and polynomial $P(x) = x^2 - 2x + 3$. Compute $P(C)$.

15) Let $D = \begin{pmatrix} 7 & 14 \\ 5 & -6 \\ -1 & 7 \end{pmatrix}$. Find $D^T$.

16) Given the matrix $A = \begin{pmatrix} 3 & -1 & 2 \\ 2 & 5 & -4 \\ -3 & 0 & 2 \end{pmatrix}$. Compute the matrix $B = -2A + 3A^T$.

17) Compute the determinant of $A = \begin{pmatrix} 3 & 2 \\ -1 & 5 \end{pmatrix}$.

18) Compute the determinant of $A = \begin{pmatrix} 0 & 1 & 2 \\ 3 & -4 & 5 \\ 7 & -1 & 4 \end{pmatrix}$.

19) Compute the determinant of $B = \begin{pmatrix} 4 & 0 & 5 & -6 \\ 0 & 5 & 7 & 4 \\ 3 & 0 & -1 & 1 \\ 0 & 1 & 4 & 1 \end{pmatrix}$.

20) Compute $\begin{vmatrix} 1+\sqrt{3} & 2-\sqrt{7} \\ 2+\sqrt{7} & 1-\sqrt{3} \end{vmatrix}$.

21) Compute $\begin{vmatrix} a-b & a \\ a+c & a+b \end{vmatrix}$.

22) Compute $\begin{vmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{vmatrix}$.

23) Solve the equation $\begin{vmatrix} 2 & x-4 \\ 1 & 4 \end{vmatrix} = 0$.

24) Solve the equation $\begin{vmatrix} 3x & -1 \\ x & 2x-3 \end{vmatrix} = \frac{3}{2}$.

25) Let $A = \begin{pmatrix} 3 & -1 \\ 5 & 6 \end{pmatrix}$, compute $A^{-1}$.

26) Let $A = \begin{pmatrix} 3 & 4 & 1 \\ 2 & -1 & 0 \\ -2 & 1 & 4 \end{pmatrix}$, compute $A^{-1}$.

27) Solve the matrix equation $X \cdot A = B$, $A = \begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix}, B = \begin{pmatrix} -1 & 2 \\ -5 & 6 \end{pmatrix}$.

28) Solve the matrix equation $A \cdot X = B$, $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, B = \begin{pmatrix} 3 & 5 \\ 5 & 9 \end{pmatrix}$.

29) Solve the matrix equation $\begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix} \cdot X \cdot \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 14 & 16 \\ 9 & 10 \end{pmatrix}$.

30) Solve the matrix equation $\begin{pmatrix} 1 & 2 & -3 \\ 3 & 2 & -4 \\ 2 & -1 & 0 \end{pmatrix} \cdot X = \begin{pmatrix} 1 & -3 & 0 \\ 10 & 2 & 7 \\ 10 & 7 & 8 \end{pmatrix}$.

31) Solve the matrix equation $\begin{pmatrix} 2 & -3 & 1 \\ 4 & -5 & 2 \\ 5 & -7 & 3 \end{pmatrix} \cdot X \cdot \begin{pmatrix} 9 & 7 & 6 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -2 \\ 18 & 12 & 9 \\ 23 & 15 & 11 \end{pmatrix}$.

32) Find the rank of the matrix $A = \begin{pmatrix} 0 & 4 & 10 & 1 \\ 4 & 8 & 18 & 7 \\ 10 & 18 & 40 & 17 \\ 1 & 7 & 17 & 3 \end{pmatrix}$.

33) Find the rank of the matrix $A = \begin{pmatrix} 1 & 2 & -3 & -5 \\ 14 & 28 & -42 & 70 \end{pmatrix}$.

34) Find the rank of the matrix $A = \begin{pmatrix} -5 & 1 & 5 & 2 & 1 \\ 6 & -2 & -10 & -4 & 1 \\ 7 & 1 & 5 & 2 & -8 \end{pmatrix}$.

35) Determine whether the system of vectors $b_1 = (1 \quad 2 \quad 2)$; $b_2 = (-1 \quad -1 \quad 0)$; $b_3 = (3 \quad 4 \quad 2)$; $b_4 = (1 \quad 3 \quad -1 \quad 0 \quad -1)$ is linearly independent.

36) Solve the system using the matrix method $\begin{cases} 2x+5y=1 \\ 3x+7y=2 \end{cases}$.

37) Solve the system using the matrix method $\begin{cases} 2x+3y+5z=10 \\ 3x+7y+4z=3 \\ x+2y+2z=3. \end{cases}$

38) Solve the system using Cramer's rule $\begin{cases} 4x_1+2x_2-x_3=1 \\ 5x_1+3x_2-2x_3=2 \\ 3x_1+2x_2-3x_3=0. \end{cases}$

39) Solve the system using Cramer's rule $\begin{cases} 5x_1+2x_2+5x_3=4 \\ 3x_1+5x_2-3x_3=-1 \\ -2x_1-4x_2+3x_3=1. \end{cases}$

40) Solve the system using Cramer's rule $\begin{cases} x_1 + x_2 + x_3 - x_4 = 3 \\ 3x_1 - 3x_2 - 2x_3 + x_4 = 6 \\ 2x_1 + 4x_2 + 3x_3 - 2x_4 = 3. \end{cases}$; $x_4 = -2, x_3 = ?$.

41) Find the general solution of the system $\begin{cases} 5x_1 + 12x_2 + 5x_3 + 3x_4 = 10 \\ 4x_1 + 3x_2 + x_3 + 3x_4 = 2 \\ 11x_1 + 11x_2 + 4x_3 + 8x_4 = 8. \end{cases}$

42) Find the general solution of the system $\begin{cases} 3x_1 + 5x_2 + x_3 - 3x_4 = 1 \\ -4x_1 - 3x_2 + x_3 + 3x_4 = -4 \\ 17x_1 + x_2 - 4x_3 - 3x_4 = 2. \end{cases}$

43) Solve the linear system

$$\begin{cases} 2x + 3y - z = 1 \\ x + 2z - y = 3 \\ 4x + y - 3z = -11. \end{cases}$$

44) Solve the linear system

$$\begin{cases} x + 2z = 4 \\ 3x - y + 4z = 9 \\ x + y - 2z = 1. \end{cases}$$

45) Solve the linear system

$$\begin{cases} x + 2y - z = 1 \\ 2x + 4y - 2z = 2 \\ x + 2y - z = 3. \end{cases}$$

46) Solve the linear system

$$\begin{cases} 2x + 4y - 2z = 8 \\ x + 2y - z = 4 \\ -3x - 6y + 3z = -12. \end{cases}$$

47) Solve the linear system

$$\begin{cases} 2x + y - z + m = 1 \\ 3x - 2y + 2z - 3m = 2 \\ 5x + y - z + 2m = -1 \\ 2x - y + z - 3m = 4 \\ 3x - m = -1. \end{cases}$$

48) Solve the linear system $\begin{cases} x + 4y - 2z + 8m = 12 \\ y - 7z + 2m = -4 \\ -m + 5z = 7 \\ z + 3m = -5. \end{cases}$

49) Find an equation involving $g, h$, and $k$ that makes this augmented matrix correspond to a consistent system

$$\left(\begin{array}{ccc|c} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{array}\right).$$

50) Determine the value of $h$ such that the matrix is the augmented matrix of a consistent linear system

$$\left(\begin{array}{cc|c} 1 & h & 4 \\ 3 & 6 & 8 \end{array}\right).$$

*Answers*

1) $A+B=\begin{pmatrix}8 & 2 & 6\\ 3 & 3 & 1\\ 0 & -4 & 2\end{pmatrix}$

2) $C$ and $G$ dimensions are not the same,; the difference between $C$and $G$ is undefined.

3) $A-B=\begin{pmatrix}-\frac{3}{4} & \frac{33}{8}\\ -\frac{1}{6} & -\frac{1}{3}\end{pmatrix}$

4) $F+K=\begin{pmatrix}34 & 75\\ -48 & -74\\ 51 & 25\end{pmatrix}$

5) $kS=\begin{pmatrix}-21 & -12 & 3\\ -6 & 0 & -12\end{pmatrix}$

6) $C=\begin{pmatrix}\frac{47}{2} & \frac{15}{2} & \frac{9}{2}\\ -\frac{5}{2} & \frac{19}{2} & -12\\ -17 & 29 & -1\end{pmatrix}$

7) $A\cdot B=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}; B\cdot A=\begin{pmatrix}-3 & -9\\ 1 & 3\end{pmatrix}$

8) $AB$ and $BA$ are both undefined because the matrix dimensions are incompatible for multiplication.

9) $AB=(-88)$

10) $AB=\begin{pmatrix}31 & -1 & 30 & -42\\ 15 & 14 & 44 & -5\\ -12 & 15 & 38 & 9\\ 53 & 24 & 64 & -22\end{pmatrix}$

11) $k=5$

12) $A^3=\begin{pmatrix}64 & -28\\ 0 & 8\end{pmatrix}$

13) $B=\begin{pmatrix}7 & -5 & 8\\ 3 & 13 & 6\\ 9 & -3 & 10\end{pmatrix}$

14) $P(C)=\begin{pmatrix}47 & 24 & -5\\ 46 & 46 & 14\\ -9 & -2 & 12\end{pmatrix}$

15) $D^T=\begin{pmatrix}7 & 5 & -1\\ 14 & -6 & 7\end{pmatrix}$

16) $B=\begin{pmatrix}3 & 8 & -13\\ -7 & 5 & 8\\ 12 & -12 & 2\end{pmatrix}$

17) $\det A=17$

18) $\det A=73$

19) $\det B=-305$

20) $1$

21) $-b^2-ac$

22) $1$

23) $x=12$

24) $x=\frac{3}{2}$ or $x=-\frac{1}{6}$

25) $A^{-1}=\frac{1}{23}\begin{pmatrix}6 & 1\\ -5 & 3\end{pmatrix}$

26) $A^{-1} = \frac{1}{44}\begin{pmatrix} 4 & 15 & -1 \\ 8 & -14 & -2 \\ 0 & 11 & 11 \end{pmatrix}$
27) $X = \begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix}$
28) $X = \begin{pmatrix} -1 & -1 \\ 2 & 3 \end{pmatrix}$
29) $X = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$
30) $X = \begin{pmatrix} 6 & 4 & 5 \\ 2 & 1 & 2 \\ 3 & 3 & 3 \end{pmatrix}$
31) $X = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$
32) $r = 2$
33) $r = 2$
34) $r = 2$
35) Linearly dependent
36) $x = 3; y = -1$
37) $x = 3, y = -2,\ z = 2$
38) $(-1,3,1)$
39) $(-7,7,5)$
40) $\left(\frac{3}{2}, -\frac{5}{2}, 2, -2\right)$
41) $\left(\frac{x_3-9x_4-2}{11}, \frac{-5x_3+x_4+10}{11}, x_3, x_4\right)$
42) $\left(\frac{-6x_4-57}{85}, \frac{69x_4+103}{85}, \frac{-72x_4-259}{85}, x_4\right)$
43) $(-1,2,3)$
44) $(2,1,1)$
45) The system is inconsistent; there is no solution.
46) There are infinitely many solutions $(4-2y+z, y, z)$.
47) The system is inconsistent; there is no solution.
48) $(2,7,1,-2)$
49) $k = -2g - h$
50) The system is consistent for all $h$, except $h = 2$.

***Exercises of applications***

1) In a certain industry, 4 factories produce 3 types of products. The $4 \times 3$ Matrix A represents the production volumes at each factory in the first quarter, and the $4 \times 3$ Matrix B, respectively, in the second quarter. The elements $(a_{ij}, b_{ij})$ represent the volumes of product type $j$ produced by factory $i$ in the first and second quarters, respectively.

$$A = \begin{pmatrix} 75 & 125 & 41 \\ 12 & 36 & 45 \\ 159 & 267 & 341 \\ 10 & 500 & 10 \end{pmatrix}, \ B = \begin{pmatrix} 15 & 99 & 21 \\ 22 & 40 & 12 \\ 100 & 213 & 196 \\ 120 & 700 & 45 \end{pmatrix}$$

*Find:* Production volumes and increase in production volumes in the second quarter compared to the first, by product types and factories.

2) Data on the harvest of three types of crops – wheat, corn, and barley (in tons) – on two farms in the years 2023 and 2024 are given in the form of matrices:

$$H_{2023} = \begin{pmatrix} 520 & 340 & 290 \\ 480 & 360 & 310 \end{pmatrix}, H_{2024} = \begin{pmatrix} 540 & 355 & 305 \\ 500 & 710 & 250 \end{pmatrix}.$$

Find: The matrix of the average annual harvest of crops.

3) An eco-processing plant has three recycling lines: 1 – paper, 2 – glass, 3 – plastic. Part of the output is used internally (to support operations and supply other lines), while the rest is the final product delivered outside the system.

Tasks:

1. Determine the distribution of products among the lines used for internal consumption.
2. Find the total output of each line, given that:

direct consumption (input coefficient) matrix

$$A = \begin{pmatrix} 0.5 & 0.2 & 0.1 \\ 0.1 & 0.4 & 0.2 \\ 0.2 & 0.1 & 0.6 \end{pmatrix},$$

And final demand vector (in tons)

$$Y = \begin{pmatrix} 80 \\ 50 \\ 70 \end{pmatrix}.$$

4) A beam is subjected to forces in three directions, and the equilibrium conditions are given by the system

$$\begin{cases} F_1 + F_2 + F_3 = 100 \\ 2F_1 - F_2 = 20 \\ F_2 + 3F_3 = 150 \end{cases}.$$

Find $F_1, F_2,$ and $F_3$.

5) An electric circuit has three loops. According to Kirchhoff's laws, the system of equations is

$$\begin{cases} 10I_1 - 2I_2 + I_3 = 5 \\ -2I_1 + 8I_2 - I_3 = 0. \\ I_1 - I_2 + 5I_3 = 10 \end{cases}$$

Find the currents $I_1, I_2,$ and $I_3$.

6) In three cities (A, B, C), residents move each year according to the following rules: 80% of the residents of A stay in A, 15% move to B, and 5% move to C. 10% of the residents of B move to A, 70% stay in B, and 20% move to C. 15% of the residents of C move to A, 5% move to B, and 80% stay in C.

The initial population (in thousands) is

$$X_0 = \begin{pmatrix} 100 \\ 50 \\ 30 \end{pmatrix}.$$

7) A pixel has the color values $(R, G, B) = (120, 200, 150)$.
To increase the brightness by 20%, the following transformation matrix is applied:

$$T = \begin{pmatrix} 1.2 & 0 & 0 \\ 0 & 1.2 & 0 \\ 0 & 0 & 1.2 \end{pmatrix}.$$

What will be the new color component values of the pixel after applying this transformation?

8) A transportation company has three bus routes (rows) and data for four days (columns):

$$A = \begin{pmatrix} 120 & 135 & 150 & 140 \\ 80 & 95 & 100 & 110 \\ 200 & 210 & 190 & 205 \end{pmatrix}.$$

Find the total number of passengers over 4 days for each route and the daily average. Model the effect of a 15% increase in passenger traffic (e.g., adding one more bus) and recalculate the totals.

## Introduction

This chapter is devoted to vectors and the basic concepts of vector algebra. Vectors provide a powerful tool for describing quantities that have both magnitude and direction, such as force, velocity, and so on. The idea of directed quantities can be traced back to antiquity: even Aristotle regarded forces as objects characterized by both magnitude and direction. However, the vector is a relatively modern mathematical concept that took shape in the second half of the nineteenth century and developed from the geometric interpretation of complex numbers. The term "vector" itself was first introduced in 1845 by the Irish mathematician, physicist, and astronomer William Rowan Hamilton (1805–1865) in his works on numerical systems generalizing complex numbers. Almost simultaneously, but from a different perspective, similar studies were carried out by the German mathematician Hermann Grassmann (1809–1877). Hamilton and Grassmann independently proposed algebraic methods for working with directed quantities. Later, the English mathematician William Kingdon Clifford (1845–1879) unified these approaches within a general theory that included vector calculus. Modern vector algebra and vector analysis were formed at the end of the nineteenth century through the works of Josiah Willard Gibbs (1839–1903) and Oliver Heaviside (1850 -1925), who independently developed a practical and convenient mathematical framework for describing the laws of electromagnetism. In 1901, Gibbs published a fundamental textbook on vector analysis.

Since then, vectors have become a basic mathematical tool in physics and engineering for describing forces, velocities, electric and magnetic fields, and other physical quantities. Vectors are also widely used in GPS navigation, computer graphics, and air and maritime navigation.

In this chapter, the definition of a vector and the basic operations on vectors are introduced. Scalar, vector, and mixed products are considered. Numerous solved examples and exercises are included to support understanding and independent practice. Examples illustrating some applications of vectors in physics, engineering, and navigation are also provided.

### 3.1. Basic concepts

There are different kinds of physical quantities. Quantities that are completely determined by their numerical value are called **scalars**. Examples of scalar quantities include area, length, volume, temperature, work, and mass. Other quantities, such as force, velocity, and acceleration, depend not only on their numerical value but also on their direction. These quantities are called **vectors**. A vector quantity is represented geometrically by a vector.

**Definition**: A **vector** is a directed line segment determined by its initial point and endpoint. In other words, a vector is a directed line segment with a starting point and an endpoint.

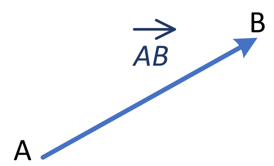



*Figure 1*. Vector

A vector is usually denoted using a lowercase letter with an arrow above it, such as $\vec{a}$, or with uppercase letters with an arrow, like $\overrightarrow{AB}$, where A is the starting point (tail of the vector), and B is the endpoint (head of the vector). Bold letters without an arrow are also used to denote vectors, such as ***a*** or **AB**.

In drawings and diagrams, vectors are shown as an arrow directed from the initial point A to the terminal point B (see Fig.1).

The **initial** point A is also called the starting point of the vector, or the origin, or the **tail** of the vector. The **endpoint** of the vector, which is represented by the arrowhead on a directed line segment, is also called the terminal point, the final point, or the **head** of the vector.

**Definition**: The **magnitude** (or norm) of a vector is the length of the directed segment AB and is denoted $|\overrightarrow{AB}|$ or $|\vec{a}|$

**Definition**: A vector whose magnitude (length) is equal to **one** is called the **unit vector**.

**Definition**: A vector whose start point and endpoint coincide (such as $\overrightarrow{AA}$ ) is called the **zero-vector** $\vec{0}$.

The magnitude of the zero vector is equal to zero: $|\vec{0}|$= 0.

**Definition**: Two vectors are called **collinear** if they lie on parallel lines or on the same line (denoted as $\vec{a}\,||\vec{b}$).

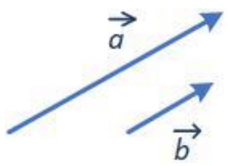



*Figure 2a*. Collinear vectors with the same direction

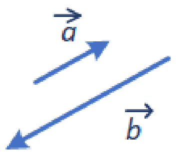



*Figure 2b*. Collinear vectors with opposite direction

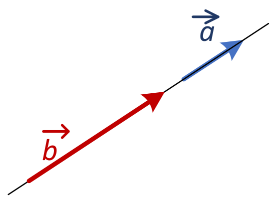



*Figure 2c*. Collinear vectors on the same line

Collinear vectors can be either in the same direction (denoted as $\vec{a} \uparrow\uparrow \vec{b}$ ) or in opposite directions ($\vec{a} \uparrow\downarrow \vec{b}$) (see Fig.2a and Fig.2b).

The zero vector is considered collinear with any vector.

**Equality of vectors**

Two **vectors** $\vec{a}$ and $\vec{b}$ are considered **equal** ($\vec{a} = \vec{b}$) if they are collinear, have the same direction ($\vec{a} \uparrow\uparrow \vec{b}$), and magnitudes ($|\vec{a}| = |\vec{b}|$).

Collinear vectors with equal magnitudes but opposite directions are called **opposite vectors** and are denoted as $\vec{a} = -\vec{b}$.

***Example**:* In Figure 3, vectors $\vec{a}$ and $\vec{b}$ are equal ( $\vec{a} = \vec{b}$ ), but vectors $\vec{c}$ and $\vec{d}$ are opposite vectors.

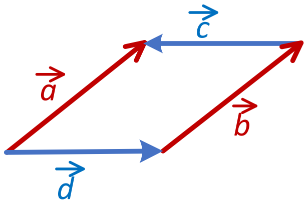



*Figure 3*.

According to the definition of vector equality, a vector may be translated parallel to itself without changing its magnitude or direction. Vectors whose physical properties remain unchanged under such a translation are called **free vectors**. In this course, only free vectors are

**Definition**: Three vectors in space are **called coplanar** vectors if they lie in the same plane or in parallel planes.

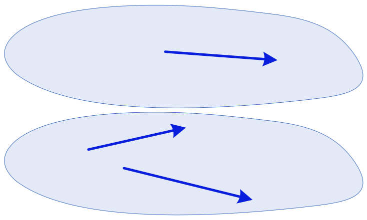


*Figure 4a*. Coplanar vectors in parallel planes

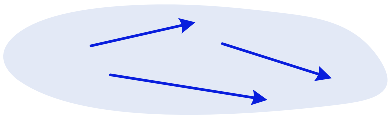


*Figure 4b*. Coplanar vectors in one plane

**3.2. Linear operations on vectors. Geometric approach**

Let $\vec{a}$ and $\vec{b}$ be two arbitrary vectors (see Fig. 5)

Linear operations on vectors include vector addition, vector subtraction, and vector multiplication by a scalar.

***2.2.1. The vector addition**:*

1) The **triangle rule**

Take an arbitrary point O and construct a vector $\overrightarrow{OA} = \vec{a}$, then, from point A, lay off the vector $\overrightarrow{AB} = \vec{b}$.

The addition of vectors $\vec{a}$ and $\vec{b}$ is the vector $\overrightarrow{OB}$, connecting the terminal point (head) of the first vector $\vec{a}$ with the initial point (tail) of the second vector $\vec{b}$ and denoted as $\vec{a} + \vec{b}$ (Fig. 6). This approach of vector addition is called the **triangle rule.**

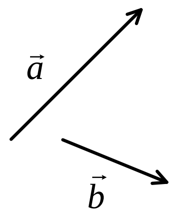



*Figure 5. Vectors*

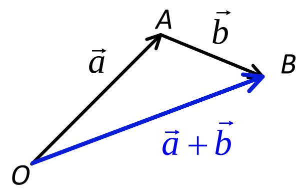



*Figure 6. Triangle rule*

**Definition (triangle rule)**: If the head of the first vector $\vec{a}$ coincides with the tail of the second vector $\vec{b}$, then the vector drawn from the tail of the vector $\vec{a}$ to the head of the vector $\vec{b}$ is called the addition of vectors $\vec{a}$ and $\vec{b}$ and is denoted $\vec{c} = \vec{a} + \vec{b}$.

The addition of two vectors can also be constructed using another approach, called the parallelogram rule.

2) the **parallelogram rule**

Take an arbitrary point O and construct vectors $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$ from point *O.* Then construct a parallelogram *OADB*, as shown in Fig.6, where vectors $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$ are

two adjacent sides of the parallelogram. The sum of vectors $\vec{a}$ and $\vec{b}$ is the vector $\overrightarrow{OB}$, which coincides with the diagonal OB of the parallelogram and is directed from vertex O to vertex D (Fig. 7).

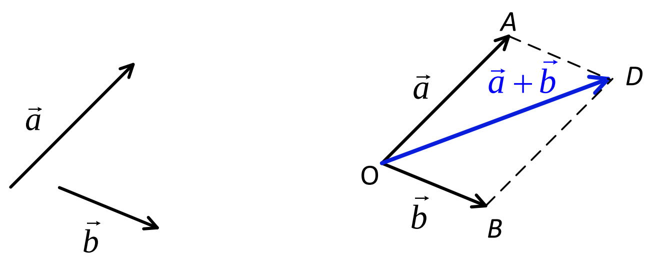



*Figure 7. Parallelogram rule*

**Definition (parallelogram rule)**: If vectors $\vec{a}$ and $\vec{b}$ originate from the same point, then their sum vector $\vec{c}$, which also starts from this common initial point and is the diagonal of the parallelogram whose sides are the vectors $\vec{a}$ and $\vec{b}$ .

3) the **polygon rule:**

To add several vectors, the **polygon rule** is used: the vectors are placed head to tail in sequence, and the resulting vector is drawn from the tail of the first vector to the head of the last vector. In Fig.7 the addition of four vectors is shown.

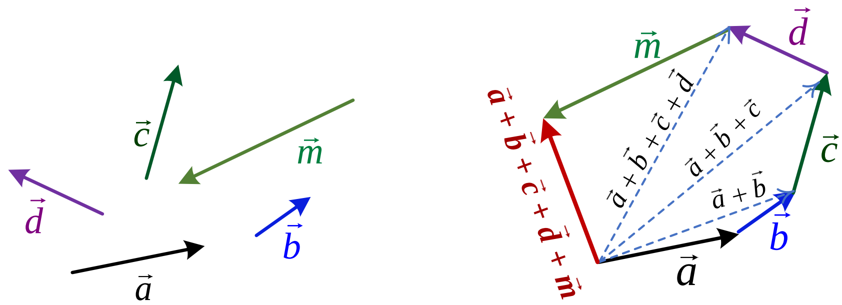



*Figure 8.* The polygon rule

Note that the polygon method can be used even to add non-coplanar vectors.

To add three non-coplanar vectors $\vec{a}$ , $\vec{b}, \vec{c}$  starting from a common initial point *O*, one constructs a parallelepiped whose three edges issuing from vertex *O* coincide with the given vectors $\overrightarrow{OA} = \vec{a}$ , $\overrightarrow{OB} = \vec{b}, \overrightarrow{OC} = \vec{c}$ (Fig. 9).

The resultant vector of their sum $\vec{d} = \vec{a} + \vec{b} + \vec{c}$  is represented by the diagonal $\overrightarrow{OD}$ of the parallelepiped, i.e. $\overrightarrow{OD} = \vec{a} + \vec{b} + \vec{c}$ Indeed, according to the parallelogram rule, we have: $\overrightarrow{OE} = \vec{a} + \vec{b}$ and $\overrightarrow{OD} = \overrightarrow{OE} + \overrightarrow{OC} = \vec{a} + \vec{b}$.

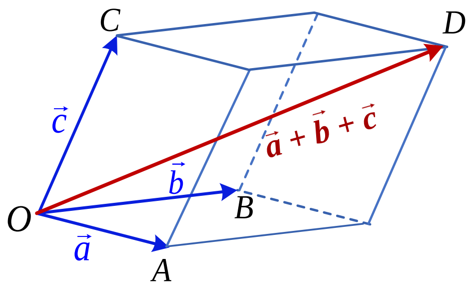



*Figure 9.* Addition of three non-coplanar vectors

**The properties of vector addition**:

1. Commutative property:

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

2. Associative property:

$$\left(\vec{a} + \vec{b}\right) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}) = \vec{a} + \vec{b} + \vec{c}$$

3. The **triangle inequality** for vectors:

The magnitude of the sum of two vectors is **less than or equal to** the sum of their magnitudes:

$$|\vec{a}+\vec{b}| \le |\vec{a}|+|\vec{b}|.$$

If vectors are not collinear, then

$$|\vec{a}+\vec{b}| < |\vec{a}|+|\vec{b}|.$$

Equality holds only if the vectors are collinear and directed in the same direction

$$|\vec{a}+\vec{b}| = |\vec{a}|+|\vec{b}| \quad \text{only if} \quad \vec{a}\uparrow\uparrow\vec{b}.$$

### *2.2.2. Vector subtraction*

**Definition:** The **subtraction** of vectors $\vec{a}$ and $\vec{b}$ is called the vector $\vec{c}$, which, when added to $\vec{b}$, yields vector $\vec{a}$ , that is $\vec{b}+\vec{c}=\vec{a}$, or $\vec{c}=\vec{a}-\vec{b}$ .

It means that vectors can be subtracted according to the following rules:

If vectors $\vec{a}$ and $\vec{b}$ originate from the same point O, the vector $\vec{c}=\vec{a}-\vec{b}$ is defined as the vector whose starting point coincides with the head of the vector point $\vec{b}$ , and the endpoint coincides with the head of the vector $\vec{a}$ (Fig. 10)

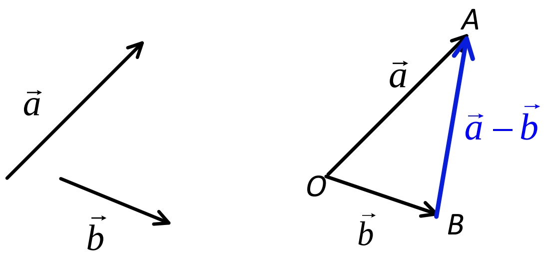



*Figure 10*. Vector subtraction

Vectors can also be subtracted according to the rule: $\vec{a}-\vec{b}=\vec{a}+(-\vec{b})$, i.e., subtraction of vectors is replaced by the addition of vectors $\vec{a}$ with the vector opposite to $\vec{b}$ (Fig. 11)

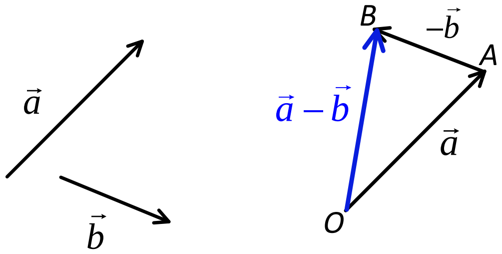


*Figure 11*. Vector subtraction as sum $\vec{a} + (-\vec{b})$

Note that in a parallelogram constructed on vectors $\vec{a}$ and $\vec{b}$, one directed diagonal is the sum of the vectors $\vec{a}$ and $\vec{b}$, while the other is their difference (Fig. 12). Here, *OADB* is the parallelogram, $\overrightarrow{OD} = \vec{a} + \vec{b}$ and $\overrightarrow{BA} = \vec{a} - \vec{b}$

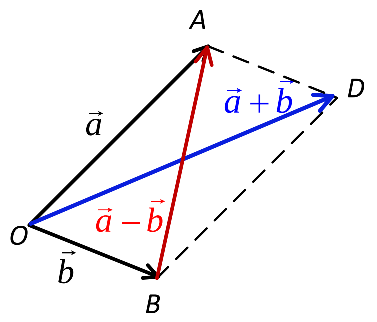



*Figure 12*. Directed diagonal in a parallelogram

### *2.2.3. Multiplication of a vector by a scalar*

**Definition:** Multiplication of a vector by a scalar *k* ( $k \neq 0$) is called the **vector** $\vec{b} = k \cdot \vec{a}$ , which

a) is collinear with the vector $\vec{a}$,

b) has magnitude $|\vec{b}| = |k| \cdot |\vec{a}|$

c) has the same direction as $\vec{a}$ ($\vec{a} \uparrow\uparrow \vec{b}$), if $k > 0$
and the opposite direction ($\vec{a} \uparrow\downarrow \vec{b}$), if $k < 0$.

***Example***: In Fig.13, one can see the given vector $\vec{a}$ and constructed vectors $\overrightarrow{CD} = 2\vec{a}$, $\overrightarrow{GH} = -3\vec{a}$ , $\overrightarrow{EF} = -\frac{1}{2}\vec{a}$ , $\overrightarrow{IJ} = -\frac{3}{4}\vec{a}$

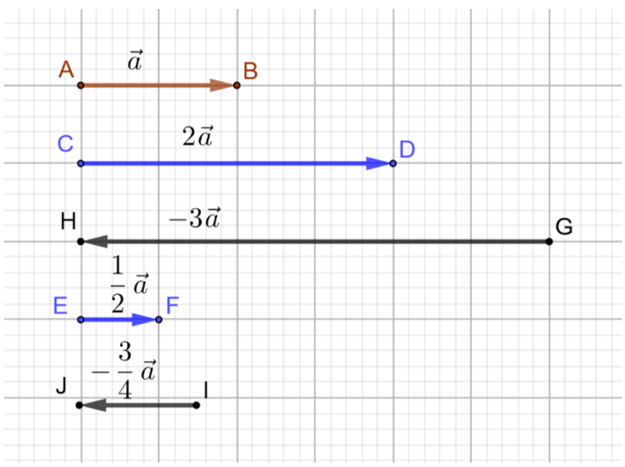



*Figure 13.* Vectors $\overrightarrow{CD} = 2\vec{a}$, $\overrightarrow{GH} = -3\vec{a}$ , $\overrightarrow{EF} = -\frac{1}{2}\vec{a}$ , $\overrightarrow{IJ} = -\frac{3}{4}\vec{a}$

**Properties of vector multiplication by a scalar**

1. Commutative property:

$$k \cdot \vec{a} = \vec{a} \cdot k$$

2. Associative property:

$$k_1 \cdot (k_2 \cdot \vec{a}) = (k_1 \cdot k_2) \cdot \vec{a}$$

3. Distributive property with respect to scalars:

$$(k_1 + k_2) \cdot \vec{a} = k_1 \cdot \vec{a} + k_2 \cdot \vec{a}$$

4. Distributive property with respect to vectors:

$$k \cdot \left(\vec{a} + \vec{b}\right) = k \cdot \vec{a} + k \cdot \vec{b}$$

From the definition of multiplication of a vector by a scalar, it follows:

If $\vec{b} = k \cdot \vec{a}$ , then $\vec{a}$ and $\vec{b}$ are collinear vectors. Conversely, if $\vec{a}$ and $\vec{b}$ are collinear vectors, then for some *k,* the equality $\vec{b} = k \cdot \vec{a}$ holds.

$$\vec{b} = k \cdot \vec{a} \quad \Leftrightarrow \quad \vec{b} || \vec{a}$$

***Example***:

Vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are shown in Figure 14.

Construct vectors:

1) $3\vec{a} + \frac{1}{2}\vec{c}$ a) using the parallelogram rule
b) using the triangle rule

2) $3\vec{a} - \frac{1}{2}\vec{c}$ 3) $3\vec{a} - \frac{1}{2}\vec{c} + \vec{b}$ 4) $-2\vec{a} + \vec{c} + \frac{2}{3}\vec{b}$

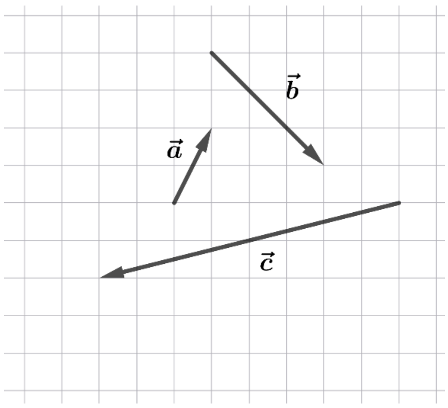


*Figure 14.* The given vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$

Solution

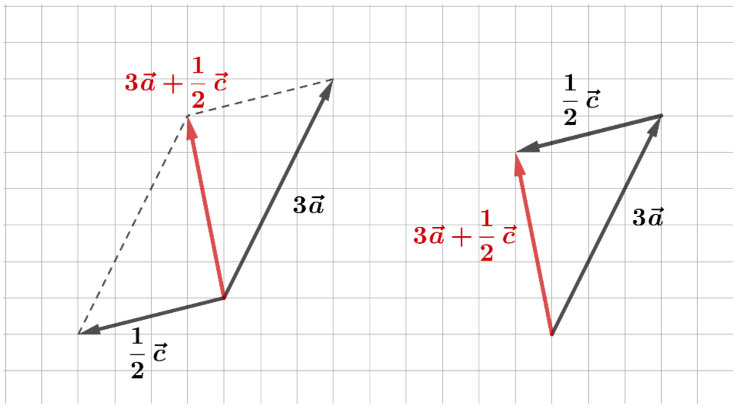



*Figure 14a.* 1) Vector $3\vec{a}+\frac{1}{2}\vec{c}$

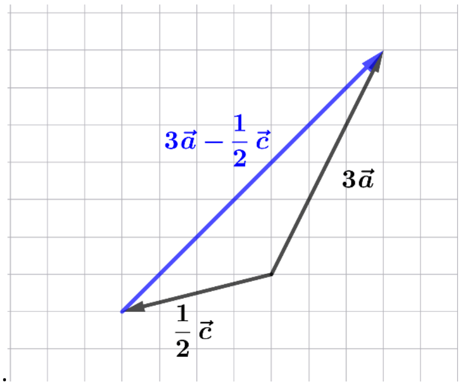



*Figure 14b.* 2) Vector $3\vec{a}-\frac{1}{2}\vec{c}$

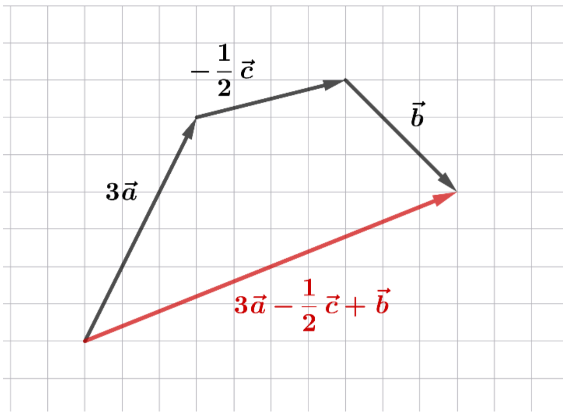



*Figure 14c.* 3) Vector $3\vec{a}-\frac{1}{2}\vec{c}+\vec{b}$

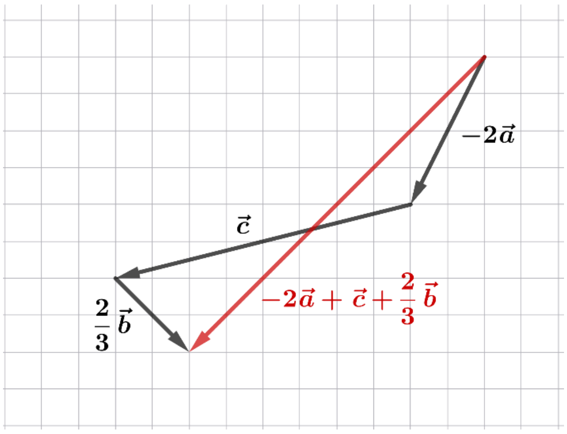



*Figure 14d.* 4) Vector $-2\vec{a}+\vec{c}+\frac{2}{3}\vec{b}$

### 3.3. Projection of a vector onto an axis

Let $l$ is given in space axis, i.e., a directed line.

**Definition:** The **projection of a point** M **onto the axis** $l$ is called the foot M’ of the perpendicular MM’ dropped from point M onto the axis. (Fig. 15).

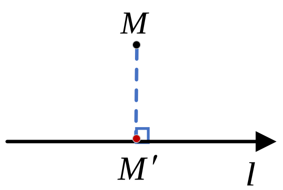



*Figure 15*. Projection of a point M onto the axis $l$

If point M lies on the axis $l$, then the projection of point M onto the axis coincides with M.

Let $\vec{a}=\overrightarrow{AB}$ is an arbitrary vector ($\overrightarrow{AB}\neq 0$) and A’ and B’ are the projections onto the axis $l$ of the initial point A and the endpoint B of the vector $\overrightarrow{AB}$. Consider the vector $\overrightarrow{A'B'}$.

**Definition:** The scalar **projection** of a vector $\vec{a} = \overrightarrow{AB}$ **onto axis** $l$ is defined as the positive number equal to the vector $\overrightarrow{A'B'}$ magnitude $\left|\overrightarrow{A'B'}\right|$ if vector $\overrightarrow{A'B'}$ and axis $l$ are oriented in the same direction, and the negative number $-\left|\overrightarrow{A'B'}\right|$ if vector $\overrightarrow{A'B'}$ and axis $l$, are oriented in opposite directions (Fig. 16). The scalar projection of the vector $\vec{a}$ onto axis $l$ can be written using different notations: $Proj_l\vec{a}$ or $a_l$ or $comp_l\vec{a}$.

$$\mathrm{Proj}_l\overrightarrow{AB} = \begin{cases} \left|\overrightarrow{A'B'}\right|, \text{if } \ \overrightarrow{A'B'} \upuparrows l \\ -\left|\overrightarrow{A'B'}\right|, \text{if } \ \overrightarrow{A'B'} \updownarrows l \end{cases}$$

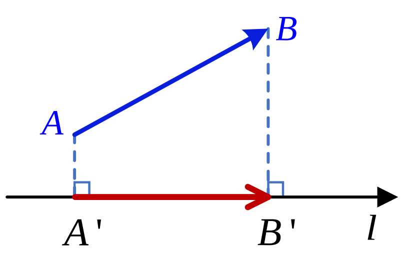



*Figure 16a.*Scalar projection onto axis $l$ is **a positive** number

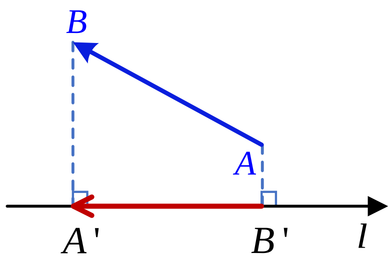



*Figure 16a.*Scalar projection onto axis $l$ is **a negative** number

If points A' and B' coincide (A'B' = 0), then the projection of the vector $\overrightarrow{AB}$ is equal to 0.

If $\overrightarrow{AB} \perp l$ then $\mathrm{Proj}_l\overrightarrow{AB} = 0$ .

The direction of the $l$-axis can be defined by a vector $\vec{l}$ lying on this axis and pointing in the same direction. Therefore, it is often said that a vector $\overrightarrow{AB}$ is projected **onto a vector** $\vec{l}$ , and this projection is denoted by $Proj_{\vec{l}}\overrightarrow{AB}$.

***Main properties of projections:***

1. The projection of the **addition** of several vectors onto the same axis is **equal to the sum of their projections** onto that axis (Fig.17).

If $\vec{c} = \vec{a} + \vec{b}$ then $Proj_l\vec{c} = Proj_l(\vec{a} + \vec{b}) = Proj_l\vec{a} + Proj_l\vec{b}$

2. When vector $\vec{a}$ is multiplied by a scalar k, its projection onto an axis is also multiplied by that number (Fig.18), i.e.

If $\vec{c} = k \cdot \vec{a}$ then $\mathrm{Proj}_l\vec{c} = \mathrm{Proj}_l(k \cdot \vec{a}) = k \cdot \mathrm{Proj}_l\vec{a}$

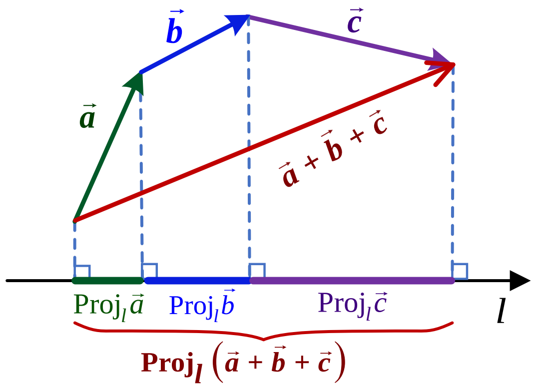



*Figure 17*. Projection of the addition of vectors

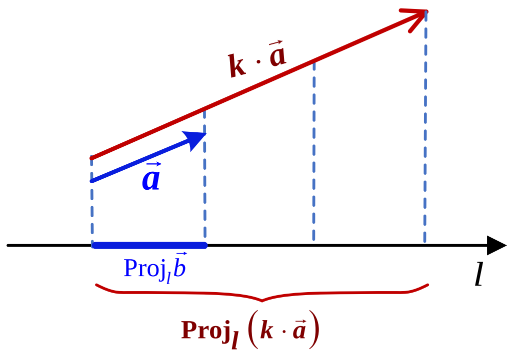



*Figure 18*. Projection of a vector multiplied by a scalar

| |
|---|
| *Thus, linear operations on vectors correspond to the same linear operations on their projections.* |

3. The projection of a vector $\vec{a}$ onto an axis $l$ (or vector $\vec{l}$ ) is equal to the product of the magnitude of the vector $\vec{a}$ and the cosine of the angle $\varphi$ between the vector and the axis (Fig.19), i.e.

$$Proj_l\vec{a} = |\vec{a}|\cdot cos\,\varphi$$

Obviously, $0 \leq \varphi \leq \pi$ . Let us prove this property.

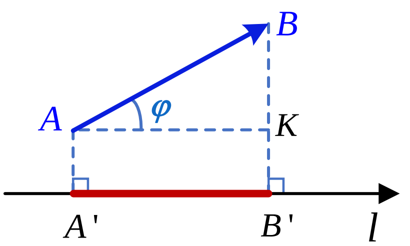



*Figure 19a.* Scalar projection onto axis $l$

$(0 < \varphi < \pi/2)$

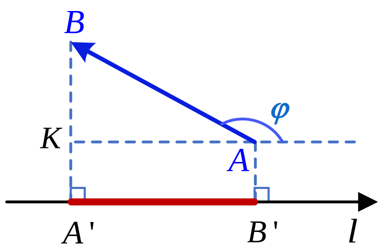



*Figure 19a.* Scalar projection onto axis $l$

$(\pi/2 < \varphi < \pi)$

***Proof:***

If the **vector forms an acute angle** with the axis, $0 < \varphi < \pi/2$ (Fig. 19 a) , then from triangle ABK (where $\angle AKB = \pi/2$ ) it follows that:

$$AK = AB \cdot cos(\angle BAK) = |\vec{a}| cos\,\varphi$$

As a result, the projection of a vector $\vec{a}$ onto the axis $l$ is

$$\text{Proj}_l\vec{a} = \left|\overrightarrow{A'B'}\right| = AK = |\vec{a}|\cdot\cos\varphi$$

If the **vector forms an obtuse** angle with the axis, $\pi/2 < \varphi < \pi$ (Fig.19b), then from triangle ABM (where $< AMB = \pi/2$ ) it follows that:

$$MA = AB \cdot cos(< BAM) = |\vec{a}| \cdot cos(180^o - \varphi) = -|\vec{a}| \cdot cos(\varphi)$$

Thus, the projection of a vector $\vec{a}$ onto the axis $l$ is

$$Proj_l\vec{a} = -\left|\overrightarrow{A'B'}\right| = -MA = |\vec{a}|\, cos(\varphi)$$

Hence, $\mathrm{Proj}_l\vec{a} = |\vec{a}| \cdot \cos\varphi$ Q.E.D.

From the last property, it follows that:

$$Proj_l\vec{a} = \begin{cases} > 0, \quad if \quad 0 < \varphi < \dfrac{\pi}{2} \ (acute\ angle) \\ < 0, \quad if \quad \dfrac{\pi}{2} < \varphi < \pi \ (obtuse\ angle) \\ = 0, \quad if \quad \varphi = \dfrac{\pi}{2} \qquad (right\ angle) \end{cases}$$

Moreover, the projections of equal vectors onto the same axis are equal.

$$\vec{a} = \vec{b} \quad \Leftrightarrow \quad Proj_l\vec{a} = Proj_l\vec{b}$$

***Example:***

The vector $\overrightarrow{AB}$ has magnitude $\left|\overrightarrow{AB}\right| = 10$ and forms an angle $\varphi = 120^0$ with axes *l*.

Find the scalar projection of the vector onto the axes $l$.

Solution

The scalar projection of a vector onto an axis is

$$Proj_l\overrightarrow{AB} = \left|\overrightarrow{AB}\right| cos\varphi = 10\ cos120^0 = 10 \cdot \left(-\frac{1}{2}\right) = -5$$

Hence, the scalar projection of the vector on the form with axes is -5.

The negative projection indicates that the angle between the vector $\overrightarrow{AB}$ and the axis $l$ is obtuse.

### 3.4. Vector Coordinates. Vector Magnitude. Direction Cosines

### 2.4.1. Decomposition of a Vector Along the Coordinate Axes. Orthogonal components of a Vector.

Let us consider the Cartesian coordinate system *Oxyz* in space, and three mutually perpendicular unit vectors $\vec{\imath}$ , $\vec{\jmath}$ , $\vec{k}$ directed along the coordinate axes *Ox*, *Oy,* and *Oz*, respectively, all originating from the same initial point (Fig. ):

$$\vec{\imath} \uparrow\uparrow Ox,\ \vec{\jmath} \uparrow\uparrow Oy\ ,\ \vec{k} \uparrow\uparrow Oz \text{ and } |\vec{\imath}| = |\vec{\jmath}| = |\vec{k}| = 1$$

$$\vec{\imath} \perp \vec{\jmath} \perp \vec{k},$$

The unit vectors $\vec{\imath}$ , $\vec{\jmath}$ , $\vec{k}$ are referred to as orthogonal **basic vectors** or **basic vectors** of the Cartesian coordinate system.

In $\mathbb{R}^3$ any vector in the Cartesian coordinate system can be uniquely decomposed into the basis vectors $\vec{\imath}$ , $\vec{\jmath}$ , $\vec{k}$ . In other words, any vector can be uniquely expressed as a linear combination of the basic vectors $\vec{\imath}$ , $\vec{\jmath}$ , $\vec{k}$ .

Let us consider an arbitrary vector $\vec{a}$ in space and move it so that its initial point coincides with the origin of the coordinate system: $\vec{a} = \overrightarrow{OM}$ (see Fig.20)

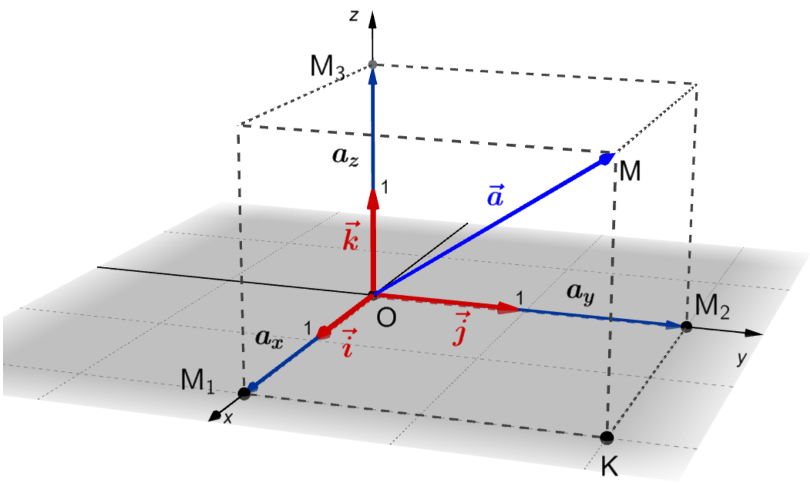



*Figure 20*. Vector $\vec{a} = \overrightarrow{OM}$ in space and its projections onto the coordinate axes.

A vector whose initial point coincides with the origin of the coordinate system is called a **radius vector**.

We will find the projections of the vector $\vec{a}$ onto the coordinate axes. To do this, we draw planes through the endpoint *M* of the vector that are parallel to the coordinate planes. The projections of *M* onto the coordinate axes are defined as the points where these planes intersect the axes – points $M_1$, $M_2$, and $M_3$. As a result, one obtains a rectangular parallelepiped, one diagonal of which is the vector $\overrightarrow{OM}$. It should be noted that straight line $M_1M$ is perpendicular to the *Ox* axis, $M_2M$ is perpendicular to the *Oy* axis, and $M_3$ *is* perpendicular to the *Oz* axis according to the **three perpendiculars theorem.**

According to the vector addition law for three vectors drawn from a common point in space, the diagonal vector of this parallelepiped can be expressed as the sum of the vectors $\overrightarrow{OM_1}$, $\overrightarrow{OM_2}$, $\overrightarrow{OM_3}$

$$\vec{a} = \overrightarrow{OM} = \overrightarrow{OM_1} + \overrightarrow{OM_2} + \overrightarrow{OM_3}$$

Since the vectors $\overrightarrow{OM_1}$ and $\vec{\imath}$ are collinear, then there exists a scalar $\lambda_1$ such that

$$\overrightarrow{OM_1} = \lambda_1 \vec{\imath}\,.$$

Similarly, the vectors $\overrightarrow{OM_2}$ un $\vec{\jmath}$ are collinear, and the vectors $\overrightarrow{OM_3}$ and $\vec{k}$ are also collinear. Therefore, there exist scalars $\lambda_2$ and $\lambda_3$, such that,

$$\overrightarrow{OM_2} = \lambda_2 \vec{\jmath} \quad , \overrightarrow{OM_3} = \lambda_3 \vec{k}\,.$$

Hence, the vector $\vec{a}$ can be expressed as a linear combination of the basis vectors $\vec{\imath}$ , $\vec{\jmath}$ , $\vec{k}$ :

$$\vec{a} = \overrightarrow{OM} = \lambda_1 \vec{\imath} + \lambda_2 \vec{\jmath} + \lambda_3 \vec{k}$$

Let us determine the geometric meaning of the coefficients $\lambda_1, \lambda_2, \lambda_3$ .

Since $\vec{\imath}$ , $\vec{\jmath}$ , $\vec{k}$ are unit vectors, i.e, $|\vec{\imath}| = |\vec{\jmath}| = |\vec{k}| = 1$, it follows that

$\overrightarrow{OM_1} = |\overrightarrow{OM_1}| \cdot \vec{\imath}$ , if vectors $\overrightarrow{OM_1}$ and $\vec{\imath}$ are directed the same way ( $\overrightarrow{OM_1} \uparrow\uparrow \vec{\imath}$ ), and

$\overrightarrow{OM_1} = -|\overrightarrow{OM_1}| \cdot \vec{\imath}$ , if the vectors $\overrightarrow{OM_1}$ and $\vec{\imath}$ are directed oppositely ( $\overrightarrow{OM_1} \uparrow\downarrow \vec{\imath}$ ).

The same reasoning applies to the vectors $\overrightarrow{OM_2}$ and $\overrightarrow{OM_3}$ :

$$\overrightarrow{OM_2} = \pm|\overrightarrow{OM_2}| \cdot \vec{\jmath}\ , \quad \overrightarrow{OM_3} = \pm|\overrightarrow{OM_3}| \cdot \vec{k}\,,$$

where the sign depends on whether the vectors $\overrightarrow{OM_2}$ and $\vec{\jmath}$ , and $\overrightarrow{OM_3}$ and $\vec{k}$ are directed in the same or in opposite directions.

This means that, according to the definition of the projection of a vector onto an axis, the coefficient $\lambda_1$ is the projection of the vector $\vec{a} = \overrightarrow{OM}$ onto the $Ox$ axis, i.e.

$$\lambda_1 = Proj_{Ox}\overrightarrow{OM} = Proj_{Ox}\vec{a}$$

Analogously, coefficients $\lambda_2$ and $\lambda_3$ are the projection of the vector onto the $Oy$ and $Oz$ axis, respectively:

$$\lambda_2 = Proj_{Oy}\overrightarrow{OM} = Proj_{Oy}\vec{a} \quad , \qquad \lambda_3 = Proj_{Oz}\overrightarrow{OM} = Proj_{Oz}\vec{a}\,.$$

If we denote the projection of the vector $\vec{a} = \overrightarrow{OM}$ onto the *Ox*, *Oy,* and *Oz* axes by $a_x$, $a_y$ and $a_z$, respectively, the vector can be uniquely expressed as a linear combination of the basic vectors $\vec{\imath}$ , $\vec{\jmath}$ , $\vec{k}$ in the following form

$$\vec{a} = a_x\vec{\imath} + a_y\vec{\jmath} + a_z\vec{k}.$$

This expression is called the **decomposition of a vector $\vec{a}$ into the basis vectors $\vec{\imath}$ , $\vec{\jmath}$ , $\vec{k}$** or decomposition of a vector along the coordinate axes, and the coefficients $a_x$, $a_y$ , $a_z$ are called the **components** (coordinates) **of the vector in the basis $\vec{\imath}$ , $\vec{\jmath}$ , $\vec{k}$** , or the Cartesian coordinates of the *vector*.

**Definition.** The *Cartesian coordinates of a vector*, or simply the coordinates of a vector, are its projections onto the corresponding coordinate axes of the Cartesian coordinate system.

The expression $\vec{a} = a_x\vec{\imath} + a_y\vec{\jmath} + a_z\vec{k}$ is often written symbolically as:

$$\vec{a} = \left(a_x; a_y; a_z\right)$$

***Example:***

Consider two vectors

$$\vec{a} = 2\vec{\imath} - \vec{\jmath} + 3\vec{k} \quad \text{and} \quad \vec{b} = 2\vec{\jmath} + \vec{k}.$$

These vectors can also be written in coordinate form as

$$\vec{a} = (2, -1, 3) \quad \text{and} \quad \vec{b} = (0, 2, 1)$$

The components of the vector $\vec{a}$ are $a_x = 2$ , $a_y = -1$ , $a_z = 3$ .

The components of the vector $\vec{b}$ are $b_x = -1$, $b_y = 0$, $b_z = 1$

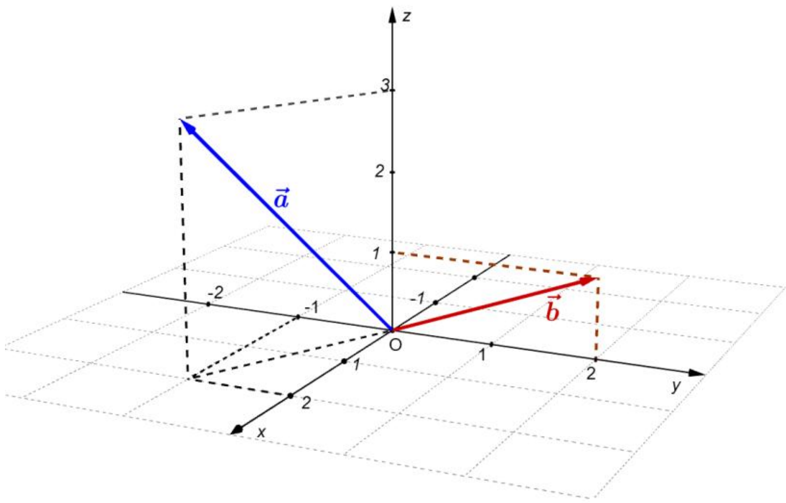



*Figure 21*. Vectors $\vec{a} = 2\vec{\imath} - \vec{\jmath} + 3\vec{k}$ and $\vec{b} = 2\vec{\jmath} + \vec{k}$

**2.4.2. Vector Magnitude. Direction Cosines**

***Vector Magnitude.***

Let $\vec{a}$ be an arbitrary vector in space. It can be translated so that its initial point coincides with the origin of the coordinate system, $\vec{a} = \overrightarrow{OM}$ (see Fig.22). Knowing the projections of the vector $\vec{a} = \overrightarrow{OM}$, an expression for its magnitude can be obtained as the length of the diagonal of the rectangular parallelepiped shown in Fig.22. For this purpose, we consider the right triangles $\triangle OM_1K$ and $\triangle OMK$, where point K is the projection of point M onto the *XOY* plane.

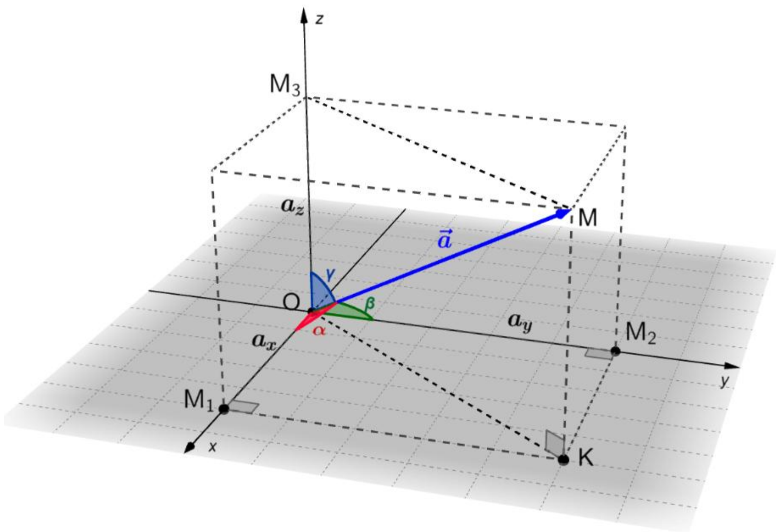



*Figure 22.*

Based on the Pythagorean theorem, one can find the hypotenuse OM of the triangle $\Delta OMK$ ($\angle K = 90^o$)

$$OM^2 = OK^2 + KM^2$$

The leg $KM$ of this triangle is equal to the edge $OM_3$ of the rectangular parallelepiped.

Consider a right triangle $\Delta OM_1K$ ($\angle K = 90^o$). In this triangle leg $M_1K$ is equal to the edge $OM_2$ of the rectangular parallelepiped. According to the Pythagorean theorem, the hypotenuse *OK* is equal to

$$OK^2 = {OM_1}^2 + M_1K^2 = {OM_1}^2 + {OM_2}^2$$

As a result, the length of the diagonal of a rectangular parallelepiped can be written:

$$OK^2 = {OM_1}^2 + {OM_2}^2 + {OM_3}^2$$

that is,

$$|\vec{a}|^2 = {a_x}^2 + {a_y}^2 + {a_z}^2$$

Hence,

$$|\vec{a}| = \sqrt{{a_x}^2 + {a_y}^2 + {a_z}^2}$$

That is, **the magnitude of a vector is equal to the square root of the sum of the squares of its projections onto the coordinate axes.**

***Direction Cosines***

Let the angles between the vector $\vec{a}$ and the axes $Ox$,$Oy$, $Oz$ be α, β, and γ, respectively (see Fig.22). The cosines of these angles are called the **direction cosines**.

Taking into account the coordinates of the vector $a_x, a_y, a_z$ are its projections onto the corresponding coordinate axes, and using the property of a vector projection onto an axis, we have:

$a_x = |\vec{a}| \cdot cos\angle(\vec{a}; Ox) = |\vec{a}| \cdot cos\alpha$ , $a_y = |\vec{a}| \cdot cos\angle(\vec{a}; Oy) = |\vec{a}| \cdot cos\beta$ ,

$a_z = |\vec{a}| \cdot cos\angle(\vec{a}; Oz) = |\vec{a}| \cdot cos\gamma$

From these relationships, it follows that the direction cosines can be determined from the formulas:

$$cos\alpha = cos\angle(\vec{a}; Ox) = \frac{a_x}{|\vec{a}|}$$
$$cos\beta = cos\angle(\vec{a}; Oy) = \frac{a_y}{|\vec{a}|}$$
$$cos\gamma = cos\angle(\vec{a}; Oz) = \frac{a_z}{|\vec{a}|}$$

**Theorem:** The sum of the squares of the direction cosines of a nonzero vector is equal to one.

$$cos^2\alpha + cos^2\beta + cos^2\gamma = 1$$

***Proof***: Consider a nonzero vector $\vec{a}$. The magnitude of the vector is

$$|\vec{a}| = \sqrt{{a_x}^2 + {a_y}^2 + {a_z}^2}$$

Let us square both parts of this expression

$$|\vec{a}|^2 = {a_x}^2 + {a_y}^2 + {a_z}^2.$$

From the definition of the direction cosines, we have:

$$a_x = |\vec{a}| \cdot \cos\alpha\ , \quad a_y = |\vec{a}| \cdot \cos\beta\ , \quad a_z = |\vec{a}| \cdot \cos\gamma$$

Substituting these expressions into the previous expression for $|\vec{a}|^2$ instead of $a_x, a_y, a_z$, we obtain:

$$|\vec{a}|^2 = |\vec{a}|^2 \cdot cos^2\alpha + |\vec{a}|^2 \cdot cos^2\beta + |\vec{a}|^2 \cdot cos^2\gamma$$

Dividing both sides by $|\vec{a}|^2$ ($|\vec{a}| \neq 0$), we obtain: $1 = cos^2\alpha + cos^2\beta + cos^2\gamma$. Q.E.D.

Note. The direction of any nonzero vector in space is completely determined by its angles with the coordinate axes.

Thus, by specifying the coordinates of a vector, one can always determine both its magnitude and direction. Conversely, if the magnitude of the vector and its direction (its directing cosines) are known, the coordinates of the vector can always be found.

***Example:***

Consider vector $\vec{a} = 2\vec{\imath} - \vec{\jmath} + 3\vec{k}$ (see Fig. 21).

The magnitude of the vector is

$$|\vec{a}| = \sqrt{{a_x}^2 + {a_y}^2 + {a_z}^2} = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{14} \approx 3.7$$

Direction cosines of the vector are:

$$cos\alpha = \frac{a_x}{|\vec{a}|}\ , \quad cos\beta = \frac{a_y}{|\vec{a}|}\ , \quad cos\gamma = \frac{a_z}{|\vec{a}|}$$

Thus,

$$cos\alpha = \frac{2}{\sqrt{14}} = \frac{\sqrt{14}}{7}\ , \ cos\beta = \frac{-1}{\sqrt{14}} = -\frac{\sqrt{14}}{14}\ , \quad cos\gamma = \frac{3}{\sqrt{14}} = \frac{3\sqrt{14}}{14}.$$

The angles between the vector $\vec{a}$ and the coordinate axes are

$$\alpha = \arccos\left(\frac{\sqrt{14}}{7}\right) \approx 57.7°$$

$$\beta = \arccos\left(-\frac{\sqrt{14}}{14}\right) \approx 105.5°$$

$$\gamma = \arccos\left(\frac{3\sqrt{14}}{14}\right) \approx 36.7°$$

### 2.4.3. Operations on Vectors in Coordinate form

Let the two vectors $\vec{a}$ and $\vec{b}$ are given by their projections on the coordinate axes $Ox$,$Oy$, $Oz$:

$\vec{a} = (a_x, a_y, a_z)$ and $\vec{b} = (b_x, b_y, b_z)$

Or equivalently

$$\vec{a} = a_x\vec{\imath} + a_y\vec{\jmath} + a_z\vec{k} \quad \text{and} \quad \vec{b} = b_x\vec{\imath} + b_y\vec{\jmath} + b_z\vec{k}$$

***Linear operations on vectors***

1) *Addition and subtraction of vectors:*

$$\vec{a} + \vec{b} = \left(a_x\vec{\imath} + a_y\vec{\jmath} + a_z\vec{k}\right) + \left(b_x\vec{\imath} + b_y\vec{\jmath} + b_z\vec{k}\right) = (a_x + b_x)\vec{\imath} + \left(a_y + b_y\right)\vec{\jmath} + (a_z + b_z)\vec{k}$$

$$\vec{a} - \vec{b} = \left(a_x\vec{\imath} + a_y\vec{\jmath} + a_z\vec{k}\right) - \left(b_x\vec{\imath} + b_y\vec{\jmath} + b_z\vec{k}\right) = (a_x - b_x)\vec{\imath} + \left(a_y - b_y\right)\vec{\jmath} + (a_z - b_z)\vec{k}$$

Or in short form:

$$\vec{a} \pm \vec{b} = \left(a_x \pm b_x;\ a_y \pm b_y;\ a_z \pm b_z\right)$$

| |
|---|
| When adding (or subtracting) vectors, their corresponding coordinates are added (or subtracted). |

2) *Multiplication by a scalar:*

$$k \cdot \vec{a} = k \cdot \left(a_x\vec{\imath} + a_y\vec{\jmath} + a_z\vec{k}\right) = k \cdot a_x\vec{\imath} + k \cdot a_y\vec{\jmath} + k \cdot a_z\vec{k}$$

$$k \cdot \vec{a} = \left(k \cdot a_x, k \cdot a_y, k \cdot a_z\right)$$

| |
|---|
| When a vector is multiplied by a scalar, the coordinates of the vector are multiplied by that scalar. |

*Example:*

Two vectors are given $\vec{a} = (2, -1, 3)$ and $\vec{b} = (-3, 15, 6)$. Find vector $\vec{c} = 2\vec{a} - \frac{1}{3}\vec{b}$ and its magnitude.

Solution

$$\vec{c} = 2\vec{a} - \frac{1}{3}\vec{b} = 2 \cdot (2, -1, 3) - \frac{1}{3} \cdot (-3, 15, 6) = (4, -2, 6) - (-1, 5, 2) =$$

$$= (4 - (-1), -2 - 5, 6 - 2) = (5, -7, 4)$$

The solution can also be written in the other form:

$$\vec{c} = 2\vec{a} - \frac{1}{3}\vec{b} = 2(2\vec{\imath} - \vec{\jmath} + 3\vec{k}) - \frac{1}{3}(-3\vec{\imath} + 15\vec{\jmath} + 6\vec{k}) =$$

$$= 4\vec{\imath} - 2\vec{\jmath} + 6\vec{k} + 1\vec{\imath} - 5\vec{\jmath} - 2\vec{k} = 5\vec{\imath} - 7\vec{\jmath} + 4\vec{k}$$

Thus,

$$\vec{c} = 2\vec{a} - \tfrac{1}{3}\vec{b} = (5, -7, 4) \quad \text{or} \quad \vec{c} = 5\vec{\imath} - 7\vec{\jmath} + 4\vec{k}$$

The magnitude of the vector $\vec{c}$ is

$$|\vec{c}| = \left|2\vec{a} - \frac{1}{3}\vec{b}\right| = \sqrt{5^2 + (-7)^2 + 4^2} = \sqrt{90} \approx 9.5$$

***Equality of Vectors***

Since a vector is a directed segment that can be moved in space parallel to itself without changing its magnitude, it follows that two vectors $\vec{a}$ and $\vec{b}$ are equal if the corresponding projections (coordinates) are equal:

$$\vec{a} = \vec{b} \quad \Leftrightarrow \quad \begin{cases} a_y = b_y \\ a_y = b_y \\ a_z = b_z \end{cases}$$

***Collinearity of Vectors***

Let vectors $\vec{a}$ and $\vec{b}$ are collinear vectors ($\vec{a}||\vec{b}$). In this case, the equality $\vec{a} = k \cdot \vec{b}$ holds.

$$\vec{a} = k \cdot \vec{b}$$

or

$$a_x\vec{\imath} + a_y\vec{\jmath} + a_z\vec{k} = k \cdot b_x\vec{\imath} + k \cdot b_y\vec{\jmath} + k \cdot b_z\vec{k}$$

It follows that

$$a_x = k \cdot b_x\,, \quad a_y = k \cdot b_y, \quad a_z = k \cdot b_z$$

or

$$\frac{a_x}{b_x} = k\,, \qquad \frac{a_y}{b_y} = k, \qquad \frac{a_z}{b_z} = k$$

In shorter form, it can be written as

$$\vec{a}||\vec{b} \iff \frac{a_x}{b_x} = \frac{a_y}{b_y} = \frac{a_z}{b_z} = k$$

Thus, the corresponding components of collinear vectors are proportional.

The converse is also true: **if the corresponding coordinates of two vectors are proportional, then the vectors are collinear**.

*Example:*

1) Consider the vectors $\vec{a} = (1, -2, 3)$ and $\vec{b} = (-3, 6, -9)$.

Consider the ratios of the corresponding components:

$$\frac{1}{-3} = \frac{-2}{6} = \frac{3}{-9} = -\frac{1}{3}$$

Since all ratios are equal, the vectors $\vec{a}$ and $\vec{b}$ are **collinear**.

The coefficient of proportionality is

$$k = -\frac{1}{3} < 0$$

It means that the vectors are **directed in opposite directions** and$\vec{a} = -\frac{1}{3}\vec{b}$ or $\vec{b} = -3\vec{a}$.

2) Let us consider the vectors $\vec{a} = (1, -2, 3)$ and $\vec{b} = (-3, 6, 9)$.

$$\frac{1}{-3} = \frac{-2}{6} \neq \frac{3}{9}.$$

The vectors are **not collinear**, since not all ratios of the corresponding components are equal.

### 2.4.4. Unit Direction Vector and its Coordinates

**Definition:** A vector with a magnitude of 1, that is colinear to the given vector $\vec{a}$ and has the same direction as the given vector, is called the **direction vector** of the vector $\vec{a}$ or the **unit vector** in the direction of the vector $\vec{a}$ and is usually denoted as $\vec{a}^0$ (or as $\hat{a}$).

$\vec{a}^0$ is the direction vector of the vector $\vec{a}$ : $\vec{a}^0 \uparrow\uparrow \vec{a}$ and $|\vec{a}^0| = 1$

This vector is also called a normalized vector.

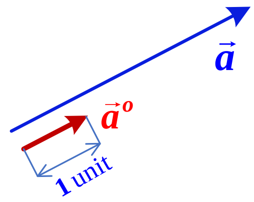



*Figure 23*. The unit vector $\vec{a}^0$ in the direction of the vector $\vec{a}$

From the definition of a unit vector, it follows that every vector can be expressed by its unit vector:

$$\vec{a} = |\vec{a}| \cdot \vec{a}^0$$

Therefore, the unit vector in the direction of $\vec{a}$ is obtained by dividing $\vec{a}$ by its magnitude:

$$\vec{a}^0 = \frac{\vec{a}}{|\vec{a}|}$$

or

$$\vec{a}^0 = \frac{(a_x, a_y, a_z)}{|\vec{a}|} = \left(\frac{a_x}{|\vec{a}|}, \frac{a_y}{|\vec{a}|}, \frac{a_z}{|\vec{a}|}\right).$$

One can see that the coordinates of the unit vector in the direction of the given vector $\vec{a}$ are equal to the direction cosines of the given vector, correspondingly.

Thus

$$\vec{a}^0 = (\cos\alpha, \cos\beta, \cos\gamma)$$

A unit vector of the given vector $\vec{a}$ as well as its direction cosines are used to indicate the direction of a vector $\vec{a}$.

***Example:***

The magnitude of the vector

$$\vec{a} = 2\vec{\imath} - 3\vec{\jmath} + 6\vec{k}$$

is

$$|\vec{a}| = \sqrt{2^2 + (-3)^2 + 6^2} = \sqrt{49} = 7.$$

The unit vector in the direction of $\vec{a}$ is

$$\vec{a}^0 = \frac{\vec{a}}{|\vec{a}|} = \frac{(2, -3, 6)}{7} = \left(\frac{2}{7}, -\frac{3}{7}, \frac{6}{7}\right)$$

## 2.4.5. Vector Coordinates with Given Initial and Terminal Point Coordinates. Length of a Segment

Let us find the coordinates of the vector $\vec{a} = \overrightarrow{AB}$ , assuming the coordinates of points $A = (x_A; y_A; z_A)$ and $B = (x_B; y_B; z_B)$ are given.

We construct two radius (position) vectors $\vec{r_1} = \overrightarrow{OA}$ and $\vec{r_2} = \overrightarrow{OB}$ ( Fig. 24).

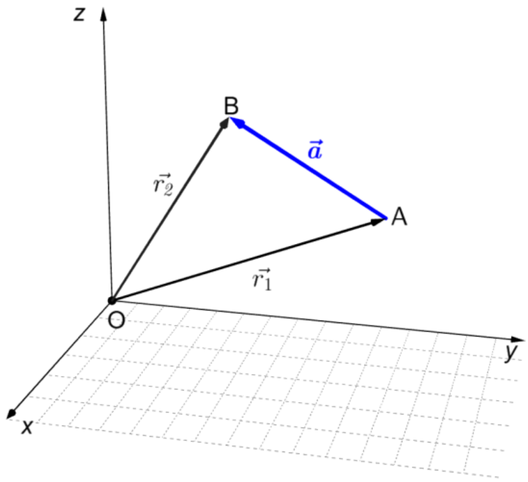



*Figure 24.* Vector $\vec{a} = \overrightarrow{AB}$ and the radius vectors $\vec{r_1} = \overrightarrow{OA}$ and $\vec{r_2} = \overrightarrow{OB}$

Since

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

and the coordinates of a position vector coincide with the coordinates of its endpoint, it follows that

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_B; y_B; z_B) - (x_A; y_A; z_A) = (x_B - x_A; y_B - y_A; z_B - z_A)$$

Thus, the coordinates of a vector are equal to the difference between the corresponding coordinates of its endpoint and start point.

Or in other words, the coordinates of a vector are obtained by subtracting the coordinates of its start point from the corresponding coordinates of its end point:

$$\overrightarrow{AB} = (x_B - \ \mathrm{x_A}; \mathrm{y}_B - \mathrm{y_A}; \mathrm{z}_B - \mathrm{z_A})$$

Note that the length of the segment AB can be found as the length of the vector $\vec{a} = \overrightarrow{AB}$

$$|\overrightarrow{AB}| = \ |\vec{a}| = \sqrt{a_x{}^2 + a_y{}^2 + a_z{}^2} = \sqrt{(\mathrm{x_B} - \mathrm{x_A})^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$$

From this, the formula for the length of the segment AB follows:

$$AB = \sqrt{(\mathrm{x_B} - \mathrm{x_A})^2 + (y_B - y_A)^2 + (y_B - y_A)^2}$$

### *2.4.6. Examples of Solved Problems:*

**Example 1**.

Given two points A(1,-2,3) and B(2,2,−1), construct the vector $\overrightarrow{AB}$. Find its coordinates, magnitude, direction cosines, and the direction vector along $\overrightarrow{AB}$.

Solution

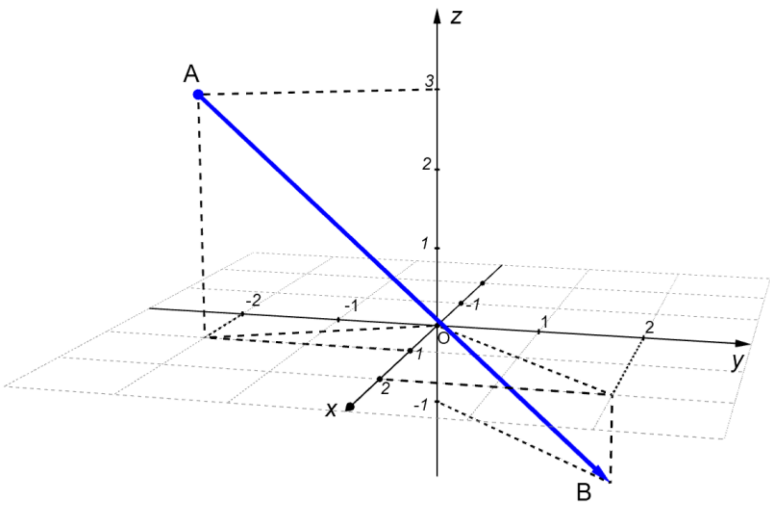



*Figure 25.* The vector $\overrightarrow{AB}$

1. Coordinates of the vector $\overrightarrow{AB}$

   $$\overrightarrow{AB} = (x_B - x_A; y_B - y_A; z_B - z_A) = (2-1, 2-(-2), -1-3) = (1,4,-4)$$

   Thus,

   $$\overrightarrow{AB} = (1,4,-4) \text{ or } \overrightarrow{AB} = 1\vec{i} + 4\vec{j} - 4\vec{k}$$

2. Magnitude of the vector is

   $$\left|\overrightarrow{AB}\right| = \sqrt{1^2 + 4^2 + (-4)^2} = \sqrt{1+16+16} = \sqrt{33} \approx 5.7$$

3. The direction vector of $\overrightarrow{AB}$ (unit vector in the direction of the vector $\overrightarrow{AB}$ ) is

   $$\overrightarrow{AB}^0 = \frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|} = \frac{(1,4,-4)}{\sqrt{21}} = \left(\frac{1}{\sqrt{33}}, \frac{4}{\sqrt{33}}, -\frac{4}{\sqrt{33}}\right) = \left(\frac{\sqrt{33}}{33}, \frac{4\sqrt{33}}{33}, -\frac{4\sqrt{33}}{33}\right)$$

   Thus,

   $$\overrightarrow{AB}^0 = \frac{\sqrt{33}}{33}\vec{i} + \frac{4\sqrt{33}}{33}\vec{j} - \frac{4\sqrt{33}}{33}\vec{k}$$

4. The direction cosines of the given vector $\overrightarrow{AB}$ coincide with corresponding components of the direction vector $\overrightarrow{AB}$ :

$$cos\alpha = \frac{\sqrt{33}}{33}\ , \qquad cos\beta = \frac{4\sqrt{33}}{33}\ , \qquad cos\gamma = -\frac{4\sqrt{33}}{33}$$

**Example 2.**

Given vector $\overrightarrow{MK} = (-2, 5, 7)$ and the terminal point $K(3;\ -4, 1)$. Find the coordinates of the initial point *M.*

Solution

Let the coordinates of point $M$ be $M(x_M, y_M, z_M)$.

The coordinates of the vector $\overrightarrow{MK}$ are obtained by subtracting the coordinates of the initial *M* point from those of the terminal point *K*:

$$\overrightarrow{MK} = (3 - x_M, -4 - y_M, 1 - z_M)$$

Since the vector $\overrightarrow{MK}$ is given, we have

$$(3 - x_M, -4 - y_M, 1 - z_M) = (-2, 5, 7)$$

Two vectors are equal if corresponding coordinates are equal:

$$3 - x_M = -2, \qquad -4 - y_M = 5\ , \qquad 1 - z_M = 7$$

Solving these equations, we obtain

$$x_M = 5, \quad y_M = -9, \quad z_M = -6$$

Thus, the coordinates of point *M* are

$$M(5, -9, -6).$$

***Example 3.***

Find the values of $\alpha$ and $\beta$ for which vectors $\vec{a} = \alpha\vec{\imath} + 3\vec{\jmath} - 2\vec{k}$ and $\vec{b} = -5\vec{\imath} + \beta\vec{\jmath} + 6\vec{k}$ are collinear.

Solution

Two vectors are collinear if their corresponding coordinates are proportional.

Therefore,

$$\frac{\alpha}{-5} = \frac{3}{\beta} = \frac{-2}{6} = k$$

From the last ratio, we find the proportionality coefficient

$$k = \frac{-2}{6} = -\frac{1}{3}$$

Using the above condition of collinearity, we find the unknowns $\alpha$ and $\beta$

$$\alpha = -5k = -5\left(-\frac{1}{3}\right) = \frac{5}{3}$$

$$\beta = \frac{3}{k} = \frac{3}{-\frac{1}{3}} = -9$$

Hence,

vectors $\vec{a} = \alpha\vec{\imath} + 3\vec{\jmath} - 2\vec{k}$ and $\vec{b} = -5\vec{\imath} + \beta\vec{\jmath} + 6\vec{k}$ are collinear for $\alpha = 5/3$ and $\beta = -9$.

***Example 4.***

Prove that the quadrilateral with vertices $A$(1,0,2),$B(4,-1,6)$,$C$(2,4,7), and $D$(-1,5,3) is a parallelogram. Find the lengths of its sides.

Solution

1) A quadrilateral is a parallelogram if one pair of its opposite sides is equal and parallel. Therefore, to prove that a quadrilateral ABCD is a parallelogram, it is sufficient to show that the vectors corresponding to one pair of its opposite sides are equal.

Compute the vectors $\overrightarrow{AB}$ and $\overrightarrow{DC}$:

$$\overrightarrow{AB} = (x_B - x_A,\ y_B - y_A,\ z_B - z_A) = (4-1, -1-0, 6-2) = (3, -1, 4)$$

$$\overrightarrow{DC} = (x_C - x_D,\ y_C - y_D,\ z_C - z_D) = (2-(-1), 4-5, 7-3) = (3, -1, 4)$$

Since $\overrightarrow{AB} = \overrightarrow{DC}$, the corresponding opposite sides AB and DC of the quadrilateral are equal and parallel. Therefore, the quadrilateral ABCD is a parallelogram.

2) To find the lengths of the parallelogram ABCD sides, we compute the magnitudes of the corresponding vectors:

$$\left|\overrightarrow{AB}\right| = \left|\overrightarrow{DC}\right| = \sqrt{3^2 + (-1)^2 + 4^2} = \sqrt{26}$$

$$\overrightarrow{BC} = (2-4, 4-(-1), 7-\ 6) = (-2, 5, 1)$$

$$\left|\overrightarrow{BC}\right| = \sqrt{(-2)^2 + 5^2 + 1^2} = \sqrt{30}$$

Thus, the lengths of the sides are

$$BC = AD = \sqrt{30}\,,\ \ AB = DC = \sqrt{26}$$

### 3.5. Products of vectors

We consider three types of products of vectors:

1) The *Scalar* (dot) *product* of two vectors. The result is a ***scalar***.
2) The *Vector* (cross) *product* of two vectors. The result is a ***vector***.
3) The *Mixed* (triple scalar) *product* of three vectors. The result is a **scalar**.

#### 2.5.1. Scalar product or Dot product of two vectors

##### 2.5.1.1. Definition of Scalar product and its properties

**Definition:** The scalar (**dot) product** of two nonzero vectors $\vec{a}$ and $\vec{b}$ is the **scalar** equal to the product of the magnitudes of these vectors and the cosine of the angle between them. It is denoted as $\vec{a} \cdot \vec{b}$ or $\vec{a}\vec{b}$.

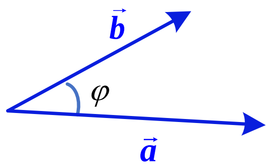



Figure 26.

The angle between two vectors is defined as the smaller angle formed when the vectors are placed with a common initial point (see Fig.26).

Thus,

$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot cos\varphi$$

where $\boldsymbol{\varphi} = \angle(\vec{a}, \vec{b})$ *and* $\mathbf{0} \le \boldsymbol{\varphi} \le \boldsymbol{\pi}$ assuming the vectors have a common initial point.

***Example****:*

Let $|\vec{a}| = \mathbf{2}$ , $|\vec{b}| = \mathbf{1}$, *and* $\angle(\vec{a}, \vec{b}) = \boldsymbol{\pi}/\mathbf{6}$, then the scalar product of the vectors is

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| cos\angle(\vec{a}, \vec{b}) = 2 \cdot 1 \cdot cos\frac{\pi}{6} = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}.$$

**Scalar product and scalar projections**:

Since the scalar projection of the vector $\vec{b}$ onto vector $\vec{a}$ is

$$proj_{\vec{a}}b = |\vec{b}| cos\angle(\vec{a}; \vec{b}) = |\vec{b}| cos\varphi$$

and the scalar projection of $\vec{\boldsymbol{a}}$ onto $\vec{\boldsymbol{b}}$ is

$$\boldsymbol{proj}_{\vec{b}}\vec{\boldsymbol{a}} = |\vec{\boldsymbol{a}}|\boldsymbol{cos}\angle(\vec{\boldsymbol{b}};\vec{\boldsymbol{a}}) = |\vec{\boldsymbol{a}}|\boldsymbol{cos\varphi}\,,$$

the scalar (dot) product can also be expressed in terms of vector projections:

$$\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{b}} = |\vec{\boldsymbol{a}}|\ \boldsymbol{proj}_{\vec{a}}\vec{\boldsymbol{b}} = |\vec{\boldsymbol{b}}|\ \boldsymbol{proj}_{\vec{b}}\vec{\boldsymbol{a}}$$

Thus, the dot product of two vectors equals the magnitude of one vector multiplied by the projection of the other onto it (onto the first vector).

**Properties of the Dot Product:**

1. *Commutative property:*

$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$$

***Proof***: $\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cdot cos\,\angle(\vec{a};\vec{b}) = |\vec{b}|\cdot|\vec{a}|\cdot cos\,\angle(\vec{b};\vec{a}) = \vec{b}\cdot\vec{a}\,.$ . Q.E.D.

2. *Associative property with respect to a scalar multiplier*:

$$(\lambda\vec{a})\cdot\vec{b} = \vec{a}\cdot(\lambda\vec{b}) = \lambda(\vec{a}\cdot\vec{b}), \qquad \lambda\,\epsilon\mathbb{R}$$

***Proof:*** Vectors $\vec{a}$ and $\lambda\vec{a}$ are collinear ($\vec{a}||\lambda\vec{a}$). It means that

a) if $\lambda > 0$, then $cos\,\angle(\lambda\vec{a};\vec{b}) = cos\,\angle(\vec{a};\vec{b})$ and

$$(\lambda\vec{a})\cdot\vec{b} = |\lambda\vec{a}|\cdot|\vec{b}|cos\,\angle(\lambda\vec{a};\vec{b}) = |\lambda|\cdot|\vec{a}|\cdot|\vec{b}|cos\,\angle(\vec{a};\vec{b}) = \lambda(\vec{a}\cdot\vec{b})$$

b) if $\lambda < 0$, then $cos\,\angle(\lambda\vec{a};\vec{b}) = cos\left(\pi - \angle(\vec{a};\vec{b})\right) = -\,cos\angle(\vec{a};\vec{b})$ and

$$(\lambda\vec{a})\cdot\vec{b} = |\lambda\vec{a}|\cdot|\vec{b}|\cdot cos\,\angle(\lambda\vec{a};\vec{b}) = -|\lambda|\cdot|\vec{a}|\cdot|\vec{b}|cos\,\angle(\vec{a};\vec{b}) = \lambda(\vec{a}\cdot\vec{b}).$$ Q.E.D.

3. *Distributive property:*

$$\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$$

***Proof**:* To prove this property, we will use the properties of the vector projection onto the other vector.

$$\vec{a}\cdot(\vec{b}+\vec{c}) = |\vec{a}|\cdot Proj_{\vec{a}}(\vec{b}+\vec{c}) = |\vec{a}|(Proj_{\vec{a}}\vec{b} + Proj_{\vec{a}}\vec{c}) =$$

$$= |\vec{a}|Proj_{\vec{a}}\vec{b} + |\vec{a}|Proj_{\vec{a}}\vec{c} = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}.$$ Q.E.D.

4. *Perpendicular vectors*

Vectors $\vec{a}$ and $\vec{b}$ are perpendicular if and only if their dot product is zero:

$$\vec{a} \cdot \vec{b} = 0 \quad \Leftrightarrow \quad \vec{a} \perp \vec{b}$$

***Proof**:* If vectors are perpendicular $\vec{a} \perp \vec{b}$ , then $\varphi = \angle(\vec{a}; \vec{b}) = 90^0$ and

$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot cos 90° \ = |\vec{a}| \cdot |\vec{b}| \cdot 0 = 0$$

Conversely, if $\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot cos\varphi \ = 0$ and $|\vec{a}| \neq 0$, $|\vec{b}| \neq 0$, then

$cos\varphi \ = 0$, so $\varphi = 90°$ and $\vec{a} \perp \vec{b}$. Q.E.D.

5. *Scalar square of a vector*

The scalar square of a vector equals the square of its magnitude:

$$\vec{a}^2 = \vec{a} \cdot \vec{a} = |\vec{a}|^2$$

***Proof***: $\vec{a}^2 = \vec{a} \cdot \vec{a} = |\vec{a}| \cdot |\vec{a}| \cdot cos 0° = |\vec{a}| \cdot |\vec{a}| \cdot 1 = |\vec{a}|^2$ Q.E.D.

***Corollary:***

The first three properties imply that the dot product of sums of vectors can be multiplied according to the standard rule for polynomial multiplication.

It follows from the 5th property that, if the square of a vector is taken and then the square root is applied, the result is the magnitude, not the vector itself ($\sqrt{\vec{a}^2} \neq \vec{a}$),

$$\sqrt{\vec{a}^2} = |\vec{a}|.$$

**Dot product** properties for **the basis vectors**:

$$\vec{\imath}^2 = \vec{\jmath}^2 = \vec{k}^2 = 1 \,, \qquad \vec{\imath} \cdot \vec{\jmath} = \vec{\imath} \cdot \vec{k} = \vec{\jmath} \cdot \vec{k} = 0$$

***Example:***

The vector $\vec{c} = 3\vec{a} + 2\vec{b}$ is given. Find the magnitude of the vector $\vec{c}$, if

$$|\vec{a}| = 2, \ |\vec{b}| = 1 \text{ and } \quad \angle(\vec{a}, \vec{b}) = 60^0$$

Solution

Since the coordinates of the vectors are not known, we use the **property of the dot product**:

$$|\vec{a}|^2 = \vec{a}^2$$

Let us find the square of the magnitude of the vector $\vec{c}$ :

$$|\vec{c}|^2 = \vec{c}^2 = \left(3\vec{a} + 2\vec{b}\right)^2 = 9\vec{a}^2 + 12\vec{a}\vec{b} + 4\vec{b}^2$$

Using properties of the dot product and the formula

$$\vec{a}\vec{b} = |\vec{a}|\,\left|\vec{b}\right| cos\,\angle\left(\vec{a},\vec{b}\right)$$

we get:

$$|\vec{c}|^2 = 9|\vec{a}|^2 + 12|\vec{a}|\left|\vec{b}\right| cos\,60^0 + 4\left|\vec{b}\right|^2 = 9 \cdot 2^2 + 12 \cdot 2 \cdot 1 \cdot \frac{1}{2} + 4 \cdot 1^2 = 52$$

Hence,

$$|\vec{c}| = \left|3\vec{a} + 2\vec{b}\right| = \sqrt{52} \approx 7.2$$

*Note*. In general,

$$\left|3\vec{a} + 2\vec{b}\right| \neq 3|\vec{a}| + 2\left|\vec{b}\right|$$

This is a common mistake made by students when replacing a vector by its magnitude.

According to the triangle inequality for vectors, the following inequality holds:

$$\left|3\vec{a} - 2\vec{b}\right| \leq 3|\vec{a}| + 2\left|\vec{b}\right|$$

#### 2.5.1.2. Dot Product in Terms of Coordinates

Let the two vectors $\vec{a}$ and $\vec{b}$ be given by their projections on the coordinate axes $Ox$,$Oy$, $Oz$:

$$\vec{a} = a_x\vec{\imath} + a_y\vec{\jmath} + a_z\vec{k}\,, \quad \vec{b} = b_x\vec{\imath} + b_y\vec{\jmath} + b_z\vec{k}$$

The dot product of can be calculated using its properties:

$$\vec{a} \cdot \vec{b} = \left(a_x\vec{\imath} + a_y\vec{\jmath} + a_z\vec{k}\right) \cdot \left(b_x\vec{\imath} + b_y\vec{\jmath} + b_z\vec{k}\right) = a_x b_x \vec{\imath}^2 + a_x b_y \vec{\imath} \cdot \vec{\jmath} + a_x b_z \vec{\imath} \cdot \vec{k} +$$

$$+ a_y b_x \vec{\jmath} \cdot \vec{\imath} + a_y b_y \vec{\jmath}^2 + a_y b_z \vec{\jmath} \cdot \vec{k} + a_z b_x \vec{k} \cdot \vec{\imath} + a_z b_y \vec{k} \cdot \vec{\jmath} + a_z b_z \vec{k}^2 =$$

$$= a_x b_x \cdot 1 + 0 + 0 + 0 + a_y b_y \cdot 1 + 0 + 0 + 0 + a_z b_z \cdot 1 = a_x b_x + a_y b_y + a_z b_z$$

Thus, the dot product of two vectors equals the sum of the products of their corresponding coordinates:

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$$

***Example**:* Two vectors are given:

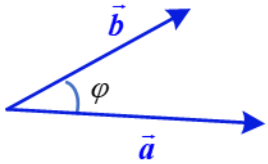
$$\vec{a} = -\vec{\imath} + 2\vec{\jmath} + 3\vec{k} \text{ and } \vec{b} = 4\vec{\imath} + 5\vec{\jmath}.$$

The dot product of the vectors is

$$\vec{a} \cdot \vec{b} = (-1, 2, 3) \cdot (4, 5, 0) = -1 \cdot 4 + 2 \cdot 5 + 3 \cdot 0 = 6.$$

**2.5.1.3. Some applications of the Scalar (Dot) product:**

*1. Finding Angle Between Two Vectors (Fig.26):*

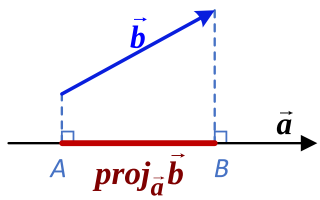



$$\cos\varphi = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|}$$

where $\varphi = \angle(\vec{a}; \vec{b})$, assumed that vectors have a common initial point.

*2. Finding the Projection of a Vector onto Another Vector (also called the scalar projection of the vector):*

$$\text{proj}_{\vec{a}}\vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}, \qquad \text{proj}_{\vec{b}}\vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|},$$




where $\text{proj}_{\vec{a}}\vec{b}$ is a projection of $\vec{b}$ onto $\vec{a}$ and $\text{proj}_{\vec{b}}\vec{a}$ is a projection of $\vec{a}$ onto $\vec{b}$.

***4.*** *Application in physics: Calculation of the Mechanical Work of a Force.*

Mechanical work is a scalar quantity that characterizes the effect of a force on a body. It depends on both the magnitude and direction of the force, as well as on the displacement of the body.

Consider a force acting on a point mass. For a constant force and rectilinear motion (straight-line motion) of a material point from point A to point B, the work done by the force is given by:

$$W = F\ s\ cos\alpha,$$

where $F$ is the magnitude of the force, $s = AB$ is the magnitude of the displacement, and α is the angle between their directions.

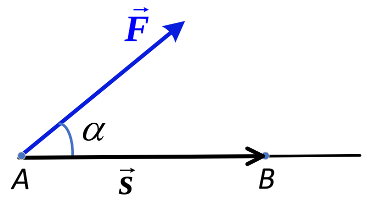



*Figure 27*. A force acting at a point

The work of the constant force can be expressed as a dot product of the force vector and the displacement :

$$W = \vec{F} \cdot \vec{s},$$

where $\vec{F}$ is the force vector and $\vec{s}$ is the displacement vector.

### 1.5.1.4. Examples of Solved Problems

***Example 1.***

Prove that the diagonals of a quadrilateral *ABCD* are mutually perpendicular, if the coordinates of the vertices are A(1; 2; 3), B(7; 3; 2), C(-3; 0; 6), D(9; 2; 4)

Solution

We form the vectors $\overrightarrow{AC}$ and $\overrightarrow{BD}$ , lying along the diagonals of the quadrilateral.

$$\overrightarrow{AC} = (-3 - 1, 0 - 2, 6 - 3) = (-4, -2, 3)$$

$$\overrightarrow{BD} = (9 - 7, 2 - 3, 4 - 2) = (2, -1, 2)$$

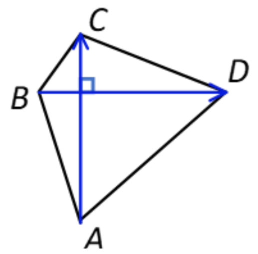



*Figure 28*. Quadrilateral *ABCD*

Compute their dot product:

$$\overrightarrow{AC} \cdot \overrightarrow{BD} = (-4, -2, 3) \cdot (2, -1, 2) = -4 \cdot 2 + (-2) \cdot (-1) + 3 \cdot 2 = -8 + 2 + 6 = 0$$

Science $\overrightarrow{AC} \cdot \overrightarrow{BD} = 0$ , vectors $\overrightarrow{AC}$ and $\overrightarrow{BD}$ are perpendiculars ($\overrightarrow{AC} \perp \overrightarrow{BD}$ ) and the diagonals of the quadrilateral *ABCD* are perpendicular.

***Example 2.***

The vertices of triangle ABC are given by:

$$A(-1, 1, 6),\ B(0, -2, 1),\ C(-4, -2, 5)$$

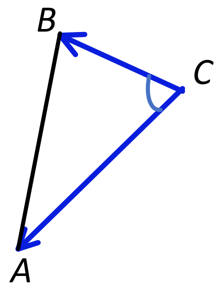



*Figure 29*. Triangle ABC and the angle at C

Find:

A) The interior angle at vertex C;

B) The median drawn from vertex C.

Solution.

*A) The interior angle at vertex C*

The interior angle at vertex C is defined as the angle between vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$ .

$$\overrightarrow{CA} = (-1-(-4), 1-(-2), 6-5) = (3,3,1)$$

$$\overrightarrow{CB} = (0-(-4), -2-(-2), 1-5) = (4,0,-4)$$

To find the angle between these vectors, we use the dot product:

$$\cos\angle C = \frac{\overrightarrow{CA}\cdot\overrightarrow{CB}}{|\overrightarrow{CA}|\cdot|\overrightarrow{CB}|}.$$

The product of the vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$ is

$$\overrightarrow{CA}\cdot\overrightarrow{CB} = (3,3,1)\cdot(4,0,-4) = 3\cdot 4 + 3\cdot 0 + 1\cdot(-4) = 12 - 4 = 8$$

Compute the magnitudes of the vectors

$$|\overrightarrow{CA}| = \sqrt{3^2 + 3^2 + 1^2} = \sqrt{19}$$

$$|\overrightarrow{CB}| = \sqrt{4^2 + 0^2 + (-4)^2} = \sqrt{32} = 4\sqrt{2}$$

Hence,

$$\cos\angle C = \frac{8}{\sqrt{19}\cdot 4\sqrt{2}} = \frac{2}{\sqrt{38}} = \frac{\sqrt{38}}{19}$$

Therefore, the interior angle at vertex C is

$$\angle C = arccos\left(\frac{\sqrt{38}}{19}\right) \approx 71.07^o.$$

*B) The determination of the median drawn from vertex C.*

Two standard methods are presented below.

*Method 1:*

We construct parallelogram ADBC based on triangle ABC (Fig. 30).

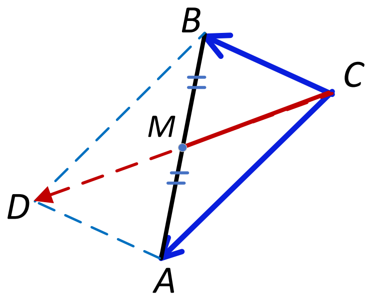



Figure 30. Triangle ABC and the median from C

The diagonals of a parallelogram intersect at their midpoint, dividing each diagonal into two equal parts. Hence, the length of the median drawn from vertex C is equal to one half of the length of the parallelogram diagonal CD, that is, one half of the magnitude of the vector $\overrightarrow{CD}$.

By the parallelogram rule for vector addition, we obtain

$$\overrightarrow{CD} = \overrightarrow{CA} + \overrightarrow{CB} = (3, 3, 1) + (4,0, -4) = (7, 3, -3)$$

The magnitude vector $\overrightarrow{CD}$ is

$$\left|\overrightarrow{CD}\right| = \sqrt{7^2 + 3^2 + (-3)^2} = \sqrt{67}$$

Consequently, the length of the median $CK$ is

$$\left|\overrightarrow{CK}\right| = \frac{\left|\overrightarrow{CD}\right|}{2} = \frac{\sqrt{67}}{2}$$

*Method 2:*

First, determine the midpoint of side AB using the formula

$$K\left(\frac{x_A + x_B}{2}, \frac{y_A + y_B}{2}, \frac{z_A + z_B}{2}\right)$$

Substituting the coordinates of points A and B, we obtain

$$K\left(\frac{-1 + 0}{2}, \frac{1 + (-2)}{2}, \frac{6 + 1}{2}\right)$$

$$K\left(-\frac{1}{2},-\frac{1}{2},\frac{7}{2}\right)$$

Next, determine vector $\overrightarrow{CK}$

$$\overrightarrow{CK} = \left(-\frac{1}{2}-(-4),-\frac{1}{2}-2, \quad \frac{7}{2}-5\right) = \left(\frac{7}{2},-\frac{3}{2},-\frac{3}{2}\right)$$

The length of the median $CK$ is equal to the magnitude of vector $\overrightarrow{CK}$

$$|\overrightarrow{CK}| = \sqrt{\left(\frac{7}{2}\right)^2+\left(-\frac{3}{2}\right)^2+\left(-\frac{3}{2}\right)^2} = \sqrt{\frac{67}{4}} = \frac{\sqrt{67}}{2} \approx 4.09$$

**Example 3.**

Vectors $\vec{a} = 2\vec{\imath}-\vec{\jmath}+2\vec{k}$ and $\vec{\mathrm{b}} = -3\vec{\imath}+2\vec{\jmath}+\vec{k}$ are given. Find the scalar projection of vector

$$\vec{\mathrm{c}} = 3\vec{b}-2\vec{a}$$

onto vector $\vec{a}$

*Solution*

First, we determine vector $\vec{\mathrm{c}}$:

$$\vec{\mathrm{c}} = 3\vec{b}-2\vec{a} = 3(-3\vec{\imath}+2\vec{\jmath}+\vec{k})-2(2\vec{\imath}-\vec{\jmath}+2\vec{k})=-13\vec{\imath}+8\vec{\jmath}-\vec{k}$$

The scalar projection of vector $\vec{\mathrm{c}}$ onto vector $\vec{a}$ is found using the formula

$$\mathrm{proj}_{\vec{a}}\vec{c} = \frac{\vec{a}\cdot\vec{\mathrm{c}}}{|\vec{a}|}$$

The scalar product of vectors $\vec{a}$ and $\vec{\mathrm{c}}$ is

$$\vec{a}\cdot\vec{\mathrm{c}} = (2,\ -1,\ 2)\cdot(-13,8,-1) = 2\cdot(-13)+(-1)\cdot 8+2\cdot(-1) = -36$$

The magnitude of vector $\vec{a}$ is

$$|\vec{a}| = \sqrt{2^2+(-1)^2+2^2} = 3.$$

Therefore, the scalar projection of vector $\vec{\mathrm{c}}$ onto vector $\vec{a}$ is

$$\mathrm{proj}_{\vec{a}}\vec{\mathrm{c}} = \frac{-36}{3} = -12$$

In this case, $\text{proj}_{\vec{a}}\vec{\text{c}} < 0$.

This means that the angle between the vectors $\vec{a}$ and $\vec{\text{c}}$ is obtuse, which ensures a negative scalar projection.

**Example 4.**

A material point is acted upon by the following forces

$$\vec{F}_1 = 2\vec{\imath} + 3\vec{\jmath} - \vec{k}\ ,\ \ \vec{F}_2 = -\vec{\imath} + 2\vec{k}\ ,\ \vec{F}_3 = 4\vec{\imath} - \vec{\jmath} + 2\vec{k}$$

Determine the work done by the resultant force when the material point moves in a straight line from point A(1, 2, 1) to point B(-1,3, 5).

*Solution*

The resultant force is the vector equal to the sum of all acting forces:

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = (2\vec{\imath} + 3\vec{\jmath} - \vec{k}) + (-\vec{\imath} + 2\vec{k}) + (4\vec{\imath} - \vec{\jmath} + 2\vec{k}) = 5\vec{\imath} + 2\vec{\jmath} + 3\vec{k}.$$

Since the motion is along a straight line from point A to point B, the displacement vector $\vec{\text{s}}$ is

$$\vec{\text{s}} = \overrightarrow{\text{AB}} = (-1 - 1, 3 - 2, 5 - 1) = (-2, 1, 4)$$

The work done by a constant force over a displacement is defined as the dot product of the force and displacement vectors:

$$\text{W} = \vec{F} \cdot \vec{\text{s}} = (5, 2, 3) \cdot (-2, 1, 4) = 5 \cdot (-2) + 2 \cdot 1 + 3 \cdot 4 = 4.$$

### 2.5.2. Vector or Cross product of two vectors

#### 2.5.2.1. Right-Handed triple and Left-Handed triple of vectors:

**Definition**. Three non-coplanar vectors $\vec{a}, \vec{b}\ , \vec{c}$ , taken in this order, form a **right-handed triple** if, when viewed from the endpoint of the third vector $\vec{c}$ , the shortest rotation from the first vector $\vec{a}$ to the second vector $\vec{b}$ appears counterclockwise. If this rotation appears clockwise, the vectors form a **left-handed triple** (Fig. 31).

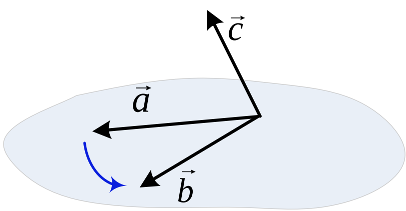



Figure 31a. Right-handed triple

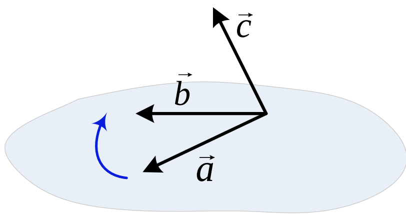



Figure 31b. Left-handed triple

A Right-hand triple $\vec{a}$, $\vec{b}$ , $\vec{c}$ can also be visualized using the right-hand rule:

If you place the index finger of your right hand along the first vector $\vec{a}$ and the middle finger along the second vector $\vec{b}$, then your thumb points in the direction of the third vector $\vec{c}$ .

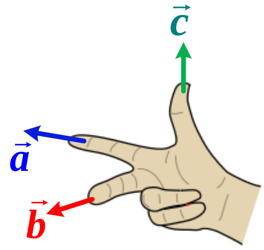



Figure 32. The direction of the cross product
*(Image adopted from work by Acdx licensed under CC-BY-SA 3.0 )*

Another explanation of the same rule:

If vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ form a right-handed triple, you can determine the direction of $\vec{c}$ by placing the fingers of your right hand along the first vector $\vec{a}$ and bending them toward the second vector $\vec{b}$ . Your thumb will point in the direction of the third vector $\vec{c}$ .

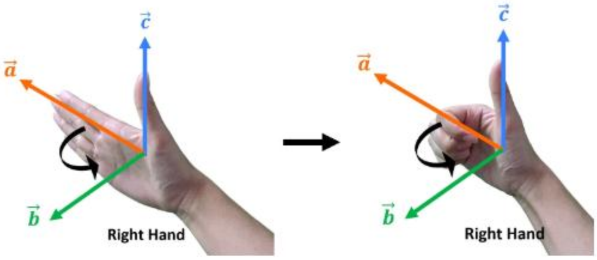



Figure 33. The direction of the cross product
*(image from https://www.wizeprep.com/online-courses )*

**Examples.**

- In the standard Cartesian coordinate system, axes $Ox, Oy, Oz$ form a right-handed triple.
- Basis unit vectors $\vec{\imath}$, $\vec{\jmath}$, $\vec{k}$ also form a right-handed triple.

**2.5.2.2. Cross (Vector) Product: Definition, Geometric Interpretation, Properties.**

**Definition:** The **Cross product** (or **vector product**) of vectors $\vec{a}$ and $\vec{b}$ is a **vector** $\vec{c}$ that satisfies the following conditions:

1. Has a magnitude

$$|\vec{c}| = |\vec{a}| \cdot |\vec{b}| \cdot \sin\varphi$$

where $\varphi = \angle(\vec{a};\vec{b})$ is the angle between vectors.

2. Is perpendicular to vectors $\vec{a}$ and $\vec{b}$

$$\vec{c} \perp \vec{a} \quad \text{and} \quad \vec{c} \perp \vec{b},$$

i.e, $\vec{c}$ is perpendicular to the plane spanned by the vectors $\vec{a}$ and $\vec{b}$ (Fig.34)

3. The vectors $\vec{a}$, $\vec{b}$ , $\vec{c}$ form a right-handed triple.

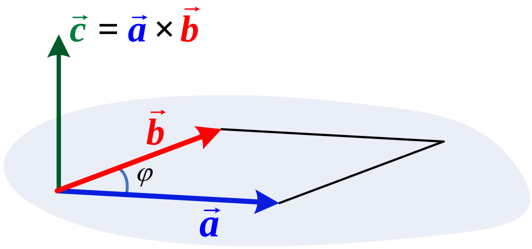



Figura 34. The cross product

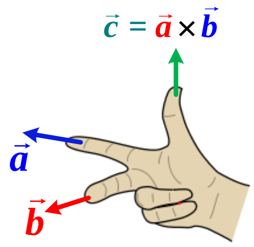



Figura 35. The direction of the cross product

( Image from work by Acdx licensed under a CC-BY-SA 3.0 license)

The vector product is denoted $\vec{a} \times \vec{b}$ or $[\vec{a},\vec{b}]$ .

From the definition, it follows that

$$|\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}| \cdot \sin\varphi$$

**Geometric interpretation of the magnitude of the cross product:**

Geometrically, the magnitude of the cross product

$$|\vec{c}| = |\vec{a} \times \vec{b}|$$

Is equal to the **area of the parallelogram** constructed on $\vec{a}$ and $\vec{b}$ (Fig. 34).

***Example***: Vectors $\vec{a}$ and $\vec{b}$ form an angle $\varphi = \pi/6$. Calculate the magnitude of their cross product $|\vec{a} \times \vec{b}|$, if $|\vec{a}| = 5$ and $|\vec{b}| = 6$.

Solution

$$|\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}| \sin\varphi = 5 \cdot 6 \cdot \sin\frac{\pi}{6} = 5 \cdot 6 \cdot \frac{1}{2} = 15$$

**Properties of the Cross Product:**

1. *Anti-Commutativity*:

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

This holds because the vectors $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$ are collinear, have the same magnitude, but point in opposite directions due to the triples $\left(\vec{a}, \vec{b}, (\vec{a} \times \vec{b})\right)$ and $\left(\vec{b}, \vec{a}, (\vec{a} \times \vec{b})\right)$ have opposite orientations.

2. *Associativity with respect to a scalar multiplier*

$$(\lambda\vec{a}) \times \vec{b} = \vec{a} \times (\lambda\vec{b}) = \lambda(\vec{a} \times \vec{b}), \qquad \lambda \in \mathbb{R}$$

***Proof:***

a) According to the definition of the cross product, both $(\lambda\vec{a}) \times \vec{b}$ and $\lambda(\vec{a} \times \vec{b})$ are perpendicular to $\vec{a}$ and $\vec{b}$ . Therefore, the vectors $(\lambda\vec{a}) \times \vec{b}$ and $\lambda(\vec{a} \times \vec{b})$ are colinear. They also have the same direction

$$(\lambda\vec{a}) \times \vec{b} \upuparrows \lambda(\vec{a} \times \vec{b}).$$

b) In addition, these vectors have the same magnitude:

$$|(\lambda\vec{a}) \times \vec{b}| = |\lambda\vec{a}| \cdot |\vec{b}| \, sin\angle(\vec{a}, \vec{b}) = |\lambda| \cdot |\vec{a}| \cdot |\vec{b}| \, sin\angle(\vec{a}, \vec{b}) = |\lambda| \cdot |(\vec{a} \times \vec{b})| =$$
$$= |\lambda(\vec{a} \times \vec{b})|$$

c) Since the vectors $(\lambda\vec{a}) \times \vec{b}$ and $\lambda(\vec{a} \times \vec{b})$ are collinear, have the same direction, and have equal magnitudes, it follows that

$$(\lambda\vec{a}) \times \vec{b} = \lambda(\vec{a} \times \vec{b}).$$

Hence, the cross product is associative with respect to scalar multiplication. Q.E.D.

3. *Distributivity*:

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

4. *Two nonzero vectors* $\vec{a}$ and $\vec{b}$ are *collinear* if and only if $\vec{a} \times \vec{b} = 0$:

$$\vec{a} \times \vec{b} = 0 \quad \Leftrightarrow \quad \vec{a} \,| \quad |\vec{b}$$

***Proof:*** If vectors $\vec{a}$ and $\vec{b}$ are collinear $\vec{a}||\vec{b}$ , then $\varphi = \angle(\vec{a}, \vec{b}) = 0°$ .

Therefore,

$$\vec{a} \times \vec{b} = |\vec{a}| \cdot |\vec{b}| \sin 0° = |\vec{a}| \cdot |\vec{b}| \cdot 0 = 0$$

Conversely, if

$$\vec{a} \times \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \sin\varphi = 0 \text{ and } |\vec{a}| \neq 0, \ |\vec{b}| \neq 0,$$

then $\sin\varphi = 0$. Therefore, $\varphi = 0°$ or $\varphi = 180°$ . It means $\vec{a}||\vec{b}$.

Hence, $\vec{a} \times \vec{b} = 0 \quad \Leftrightarrow \quad \vec{a} \,||\, \vec{b}$. Q.E.D.

5. It follows from the fourth property of the cross product that

$$\vec{a} \times \vec{a} = 0.$$

**Cross Product** properties for **the Basis Vectors**:

$$\vec{\imath} \times \vec{\imath} = \vec{\jmath} \times \vec{\jmath} = \vec{k} \times \vec{k} = 0$$

$$\vec{\imath} \times \vec{\jmath} = \vec{k}\,, \quad \vec{\jmath} \times \vec{k} = \vec{\imath}\,, \quad \vec{k} \times \vec{\imath} = \vec{\jmath}$$

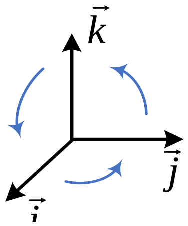


*Figure 36*. Basis Vectors

***Proof:*** These relations follow from the fact that $\vec{\imath}, \vec{\jmath}, \vec{k}$ form a right-handed orthonormal basis in $\mathbb{R}^3$.

We prove, for example, that $\vec{\imath} \times \vec{\jmath} = \vec{k}$:

a) The vector $\vec{k}$ is perpendicular to both $\vec{\imath}$ and $\vec{\jmath}$:

$$\vec{k} \perp \vec{\imath} \text{ and } \vec{k} \perp \vec{\jmath}$$

b) $|\vec{k}| = 1$ and $|\vec{\imath} \times \vec{\jmath}| = |\vec{\imath}| \cdot |\vec{\jmath}| \sin 90^o = 1$

c) Vectors $\vec{i}$, $\vec{j}$, $\vec{k}$ form a right-handed triple

Therefore, by the definition of the cross product, we conclude that $\vec{k} = \vec{i} \times \vec{j}$ .

***Example:***

Simplify the expression

$$(2\vec{a} + 5\vec{b}) \times (\vec{a} - 3\vec{b})$$

and find its magnitude

$$\left|(2\vec{a} + 5\vec{b}) \times (\vec{a} - 3\vec{b})\right|,$$

if $|\vec{a}| = 3$ , $|\vec{b}| = 2$ $and\ \angle(\vec{a}, \vec{b}) = 30^0$.

Solution

1) First, expand the cross product using distributivity:

$$(2\vec{a} + 5\vec{b}) \times (\vec{a} - 3\vec{b}) = (2\vec{a}) \times \vec{a} + (2\vec{a}) \times \left(-3\vec{b}\right) + \left(5\vec{b}\right) \times \vec{a} + \left(5\vec{b}\right) \times \left(-3\vec{b}\right) =$$

$$= 2\vec{a} \times \vec{a} - 6\vec{a} \times \vec{b} + 5\vec{b} \times \vec{a} - 15\vec{b} \times \vec{b}$$

According to the properties of the cross product,

$$\vec{a} \times \vec{a} = 0, \quad \vec{b} \times \vec{a} = -\vec{a} \times \vec{b} , \qquad \vec{b} \times \vec{b} = 0$$

Therefore,

$$(2\vec{a} + 5\vec{b}) \times (\vec{a} - 3\vec{b}) = -6\vec{a} \times \vec{b} - 5\vec{a} \times \vec{b} = -11\vec{a} \times \vec{b}$$

2) Magnitude $\left|(2\vec{a} + 5\vec{b}) \times (\vec{a} - 3\vec{b})\right|$:

The magnitude of the cross product $\vec{a} \times \vec{b}$ is

$$\left|\vec{a} \times \vec{b}\right| = |\vec{a}|\,\left|\vec{b}\right| \sin\angle\left(\vec{a}, \vec{b}\right) = 3 \cdot 2 \cdot \frac{1}{2} = 3$$

Therefore,

$$\left|(2\vec{a} + 5\vec{b}) \times (\vec{a} - 3\vec{b})\right| = \left|-11\vec{a} \times \vec{b}\right| = 11\left|\vec{a} \times \vec{b}\right| = 11 \cdot 3 = 33$$

Hence,

$(2\vec{a}+5\vec{b})\times(\vec{a}-3\vec{b}) = -11\vec{a}\times\vec{b}$ and $\left|(2\vec{a}+5\vec{b})\times(\vec{a}-3\vec{b})\right| = 33.$

***Example:***

Simplify the given expression

$$(\vec{\imath}+\vec{\jmath})\times(\vec{\jmath}+\vec{k})-\vec{\jmath}\times(\vec{\imath}+\vec{k})+\vec{k}\times(\vec{\imath}+\vec{\jmath}+\vec{k})$$

Solution

We expand the cross product using distributivity and simplify the obtained expression using cross product properties for the basis vectors:

$$(\vec{\imath}+\vec{\jmath})\times(\vec{\jmath}+\vec{k})-\vec{\jmath}\times(\vec{\imath}+\vec{k})+\vec{k}\times(\vec{\imath}+\vec{\jmath}+\vec{k})=$$

$$=\vec{\imath}\times\vec{\jmath}+\vec{\imath}\times\vec{k}+\vec{\boldsymbol{j}}\times\vec{\boldsymbol{j}}+\vec{\jmath}\times\vec{k}-\vec{\jmath}\times\vec{\imath}-\vec{\jmath}\times\vec{k}+\vec{k}\times\vec{\imath}+\vec{k}\times\vec{\jmath}+\vec{\boldsymbol{k}}\times\vec{\boldsymbol{k}}=$$

$$=\vec{\imath}\times\vec{\jmath}-\vec{k}\times\vec{\imath}+\mathbf{0}+\vec{\imath}\times\vec{\jmath}+\vec{k}\times\vec{\imath}-\vec{\jmath}\times\vec{k}+\mathbf{0}=2\vec{\imath}\times\vec{\jmath}-\vec{\jmath}\times\vec{k}=2\vec{k}-\vec{\imath}$$

Therefore,

$$(\vec{\imath}+\vec{\jmath})\times(\vec{\jmath}+\vec{k})-\vec{\jmath}\times(\vec{\imath}+\vec{k})+\vec{k}\times(\vec{\imath}+\vec{\jmath}+\vec{k})=\mathbf{2}\vec{\boldsymbol{k}}-\vec{\boldsymbol{\imath}}$$

### 2.5.2.3. Cross Product in Terms of Coordinates

Let the two vectors $\vec{a}$ and $\vec{b}$ be given by their projections on the coordinate axes $Ox$,$Oy$, $Oz$:

$$\vec{a}=a_x\vec{\imath}+a_y\vec{\jmath}+a_z\vec{k}\,,\quad \vec{b}=b_x\vec{\imath}+b_y\vec{\jmath}+b_z\vec{k}$$

The cross product can be calculated using its properties:

$$\vec{a}\times\vec{b}=(a_x\vec{\imath}+a_y\vec{\jmath}+a_z\vec{k})\times(b_x\vec{\imath}+b_y\vec{\jmath}+b_z\vec{k})=a_xb_x(\vec{\imath}\times\vec{\imath})+a_xb_y(\vec{\imath}\times\vec{\jmath})+$$

$$+a_xb_z(\vec{\imath}\times\vec{k})+a_yb_x(\vec{\jmath}\times\vec{\imath})+a_yb_y(\vec{\jmath}\times\vec{\jmath})+a_yb_z(\vec{\jmath}\times\vec{k})+a_zb_x(\vec{k}\times\vec{\imath})+a_zb_y(\vec{k}\times\vec{\jmath})+$$

$$+a_zb_z(\vec{k}\times\vec{k})=0+a_xb_y\vec{k}-a_xb_z\vec{\jmath}-a_yb_x\vec{k}+0+a_yb_z\vec{\imath}+a_zb_x\vec{\jmath}-a_zb_y\vec{\imath}+0=$$

$$=(a_yb_z-a_zb_y)\vec{\imath}-(a_xb_z-a_zb_x)\vec{\jmath}+(a_xb_y-a_yb_x)\vec{k}$$

Hence,

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} \vec{\imath} - \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} \vec{\jmath} + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \vec{k}$$

The resulting expression corresponds to the expansion of a third-order determinant along its first row. Hence, it can be rewritten in the following compact form:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

***Example:***

Two vectors are given: $\vec{a} = -\vec{\imath} + 2\vec{\jmath} + 3\vec{k}$ and $\vec{b} = 4\vec{\imath} + 5\vec{\jmath}$.

Find their cross product and its magnitude.

Solution

1. The cross product of the given vectors is the vector $\vec{c} = \vec{a} \times \vec{b}$ :

$$\vec{c} = \vec{a} \times \vec{b} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -1 & 2 & 3 \\ 4 & 5 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 5 & 0 \end{vmatrix} \vec{\imath} - \begin{vmatrix} -1 & 3 \\ 4 & 0 \end{vmatrix} \vec{\jmath} + \begin{vmatrix} -1 & 2 \\ 4 & 5 \end{vmatrix} \vec{k} =$$

$$= (0 - 15)\vec{\imath} - (0 - 12)\vec{\jmath} + (-5 - 8)\vec{k} = -15\vec{\imath} + 12\vec{\jmath} - 13\vec{k}$$

Thus,

$$\vec{c} = \vec{a} \times \vec{b} = -15\vec{\imath} + 12\vec{\jmath} - 13\vec{k}$$

2. The magnitude of the obtained vector is

$$|\vec{c}| = |\vec{a} \times \vec{b}| = \sqrt{(-15)^2 + 12^2 + (-13)^2} = \sqrt{408} \approx 20{,}2$$

### 2.5.2.4. Some Applications of the Vector Product

***Geometry***

1. *Finding the Area of a Parallelogram and a Triangle:*

   Geometrically, the magnitude $|\vec{a} \times \vec{b}|$ equals the area of the parallelogram constructed on vectors $\vec{a}$ and $\vec{b}$ (see Fig.37)

$$S_{\text{parallelogram}} = |\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}| \sin\varphi$$

   where $\varphi = \angle(\vec{a}; \vec{b})$ is the angle between vectors.

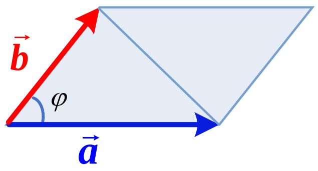



*Figure 37*. Parallelogram constructed on vectors $\vec{a}$ and $\vec{b}$ .

Thus, the area of a triangle constructed on vectors $\vec{a}$ and $\vec{b}$ is

$$S_{\Delta} = \frac{1}{2}\left|\vec{a} \times \vec{b}\right|.$$

***Physics***

1. *Finding the Moment of a Force Relative to a Point*

In physics, the moment of a force (torque) relative to a point is a vector quantity that characterizes the force's rotational effect.

Let a force $\vec{F} = \overrightarrow{AB}$ be applied at point A, and let O be an arbitrary point in space.

The moment of force F relative to point O is a vector M, which:

a) is perpendicular to the plane passing through points O, A, B;

b) has a magnitude equal to the product of the force and the lever arm

$$\left|\vec{M}\right| = \left|\vec{F}\right| \cdot OK = \left|\vec{F}\right| \cdot |\vec{r}|\sin\varphi = \left|\vec{F}\right| \cdot \left|\overrightarrow{OA}\right|\sin\angle\left(\vec{F}, \overrightarrow{OA}\right);$$

c) forms a right-handed triple with vectors $\overrightarrow{OA}$ and $\overrightarrow{AB}$.

Therefore,

$$\vec{M} = \overrightarrow{OA} \times \vec{F}$$

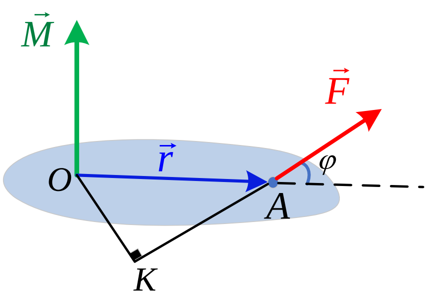



*Figure 38*. The moment of a force $\vec{F}$ relative to a point *O*.

2. *Finding Linear Velocity in Circular Motion*

In physics, the linear velocity of a point in circular motion is given by

$$\vec{V} = \vec{\omega} \times \vec{r}$$

where $\vec{\omega}$ is the angular velocity,

$\vec{r}$ is the position vector of the point measured from the center of rotation (or from the origin of the chosen reference frame)

3. *Finding the Lorentz force*

A charged particle with charge $q$, moving with velocity $\vec{V}$ in a magnetic field $\vec{B}$ experiences the magnetic Lorentz force with magnitude

$$F = qVB\sin\theta$$

where $V = |\vec{V}|$, $B = |\vec{B}|$ and $\theta = \angle(\vec{V}; \vec{B})$.

The Lorentz force is perpendicular to both the velocity vector $\vec{V}$ and the magnetic field vector $\vec{B}$, and its direction is determined by the right-hand rule for positive charges (Fig.39)

In vector form,

$$\vec{F} = q(\vec{V} \times \vec{B}).$$

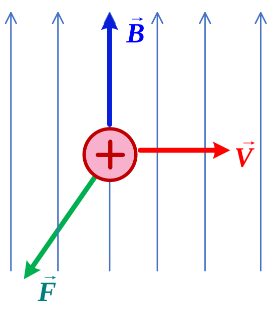


*Figure 39*. Lorentz force.

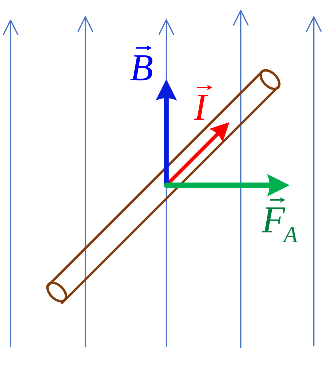


*Figure 40*. Ampere force.

6. *Finding the Ampere force*

A current-carrying conductor placed in a magnetic field $\vec{B}$ experiences the **Ampere force**. The direction of the Ampere force is determined by the left-hand rule (see Fig.40).

Its magnitude depends on the electric current, the length of the conductor, and the magnetic field intensity, and is equal

$$F = L \cdot I \cdot B \cdot sin\alpha$$

where $B$ —magnetic flux density, I — the electric current in the conductor, L — the length of the conductor in the magnetic field, $\alpha = \angle(\vec{I};\vec{B})$ is the angle between the direction of the current and the magnetic field vector.

The expression for the Ampere force in vector form is

$$\vec{F} = L(\vec{I} \times \vec{B})$$

where $\vec{I} = I \cdot \vec{e}$ , $\vec{e}$ is a unit vector along the current-carrying conductor

### 2.5.2.5. Examples of Solved Problems

***Example 1.***

The parallelogram is constructed on vectors $\vec{a} = 2\vec{\imath} + 4\vec{\jmath} - \vec{k}$ and $\vec{b} = -3\vec{\jmath} + 2\vec{k}$ . Find the area of the parallelogram and the lengths of its diagonals.

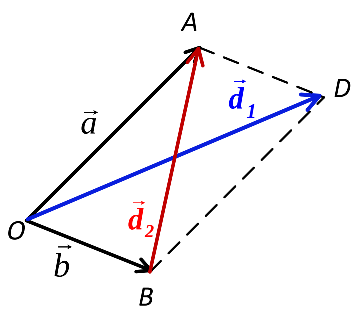



*Figure 41*. A parallelogram constructed on $\vec{a}$ and $\vec{b}$.

**Solution**

1) The area of a parallelogram constructed on two vectors is equal to the magnitude of their cross product:

$$S_{\text{parallelogram}} = |\vec{a} \times \vec{b}|$$

First, find the cross product of the vectors

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{\imath} & \vec{\jmath} & \vec{k}\\ 2 & 4 & -1\\ 0 & -3 & 2\end{vmatrix}=\begin{vmatrix}4 & -1\\ -3 & 2\end{vmatrix}\vec{\imath}-\begin{vmatrix}2 & -1\\ 0 & 2\end{vmatrix}\vec{\jmath}+\begin{vmatrix}2 & 4\\ 0 & -3\end{vmatrix}\vec{k}=$$

$$=(8-3)\vec{\imath}-(4-0)\vec{\jmath}+(-6-0)\vec{k}=5\vec{\imath}-4\vec{\jmath}-6\vec{k}$$

Hence, the area of the parallelogram is

$$S_{\text{parallelogram}}=\left|\vec{a}\times\vec{b}\right|=\sqrt{5^2+(-4)^2+(-6)^2}=\sqrt{77}$$

2) Lengths of its diagonals

The diagonals of the parallelogram correspond to the vectors $\overrightarrow{d_1}$ and $\overrightarrow{d_2}$ (Fig. 41):

$$\overrightarrow{d_1}=\vec{a}+\vec{b},\qquad \overrightarrow{d_2}=\vec{a}-\vec{b}.$$

Find the coordinates of $\overrightarrow{d_1}$ and its magnitude.

$$\overrightarrow{d_1}=\vec{a}+\vec{b}=\left(2\vec{\imath}+4\vec{\jmath}-\vec{k}\right)+\left(-3\vec{\jmath}+2\vec{k}\right)=2\vec{\imath}+\vec{\jmath}+\vec{k}$$

$$\left|\overrightarrow{d_1}\right|=\sqrt{2^2+1^2+1^2}=\sqrt{6}$$

The coordinates and the magnitude of the second vector $\overrightarrow{d_2}$ are

$$\overrightarrow{d_2}=\vec{a}-\vec{b}=\left(2\vec{\imath}+4\vec{\jmath}-\vec{k}\right)-\left(-3\vec{\jmath}+2\vec{k}\right)=2\vec{\imath}+7\vec{\jmath}-3\vec{k}$$

$$\left|\overrightarrow{d_2}\right|=\sqrt{4+49+9}=\sqrt{62}$$

Hence, the diagonal lengths are

$$d_1=\sqrt{6}\ \text{ and }\ d_2=\sqrt{62}$$

***Example 2.***

The coordinates of the vertices of triangle ABC are A(-2, 1, 4), B(-3, 0, 3), C(5, 2, 6) .

Find the area of the triangle and the length of the altitude drawn from vertex B (Fig.42).

**Solution:**

1. The area of the triangle ABC can be found using the cross product of vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ :

$$S_\Delta = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|.$$

First, we find the coordinates of these vectors

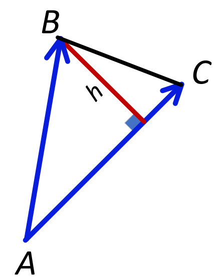


*Figure 42*. Triangle ABC from Example 2

$$\overrightarrow{AB} = (-3-(-2), 0-1, 3-4) = (-1,-1,-1)$$

$$\overrightarrow{AC} = (5-(-2), 2-1, 6-4) = (7,1,2)$$

Compute the cross product:

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -1 & -1 & -1 \\ 7 & 1 & 2 \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix}\vec{\imath} - \begin{vmatrix} -1 & -1 \\ 7 & 2 \end{vmatrix}\vec{\jmath} + \begin{vmatrix} -1 & -1 \\ 7 & 1 \end{vmatrix}\vec{k} =$$

$$= (-2+1)\vec{\imath} - (-2+7)\vec{\jmath} + (-1+7)\vec{k} = -\vec{\imath} - 5\vec{\jmath} + 6\vec{k}$$

The magnitude of the cross product is

$$\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \sqrt{(-1)^2 + (-5)^2 + 6^2} = \sqrt{62}.$$

Hence, the area of triangle ABC is

$$S_\Delta = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{\sqrt{62}}{2}$$

2. The altitude drawn from vertex B:

The altitude *h* drawn from vertex B is the perpendicular drawn from B to the opposite side AC.

The area of a triangle is equal to one half of the product of the height and the length of the base (in our case AC):

$$S_\Delta = \frac{1}{2} h \cdot AC$$

From this formula, it follows that

$$h = \frac{2S_\Delta}{AC}$$

Compute the length of side AC:

$$AC = \left|\overrightarrow{AC}\right| = \sqrt{7^2 + 1^2 + 2^2} = \sqrt{54}$$

Therefore, the altitude drawn from vertex B is equal to

$$h = \frac{2S_\Delta}{AC} = \frac{2\sqrt{62}}{2\sqrt{54}} = \sqrt{\frac{62}{54}} \approx 1.14$$

***Example 4***.

Let the vectors

$$\overrightarrow{OA} = 3\vec{p} + 2\vec{q} \quad \text{and} \quad \overrightarrow{OB} = \vec{p} - 4\vec{q}$$

Define two adjacent sides of a parallelogram.

Find the area of the parallelogram and the length of its diagonals, given that

$$|\vec{p}| = 2, |\vec{q}| = 1 \text{ and } \angle(\vec{p}; \vec{q}) = \frac{\pi}{3}.$$

Solution:

In this problem, a schematic representation of the parallelogram, shown in Fig.43a, is sufficient. However, a more accurate construction of the parallelogram based on the given vectors is shown in Fig. 43b.

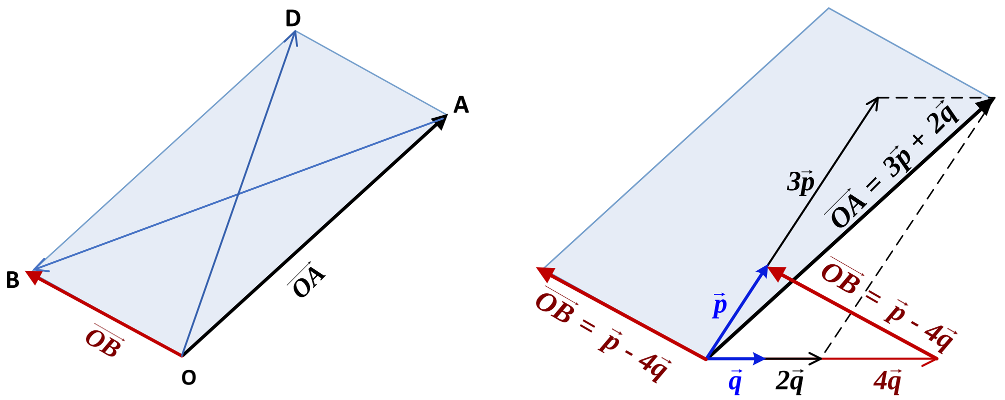



*Figure 43a*. A **schematic** representation of the parallelogram constructed on $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

*Figure 43b*. **Accurate construction** of the parallelogram on vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

1) The area S of the parallelogram constructed on the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ is equal to the magnitude of their cross product:

$$S = \left|\overrightarrow{OA} \times \overrightarrow{OB}\right|$$

First, compute the cross product $\overrightarrow{OA} \times \overrightarrow{OB}$ , by substituting the given vector expressions and simplifying the result:

$$\overrightarrow{OA} \times \overrightarrow{OB} = (3\vec{p} + 2\vec{q}) \times (\vec{p} - 4\vec{q}) = 3\vec{p} \times \vec{p} - 12\vec{p} \times \vec{q} + 2\vec{q} \times \vec{p} - 8\,\vec{q} \times \vec{q}$$

Using the properties of the cross product

$$\vec{p} \times \vec{p} = 0, \qquad \vec{q} \times \vec{q} = 0 \ , \ \ \vec{q} \times \vec{p} = -\,\vec{p} \times \vec{q}\ ,$$

the expression simplifies to

$$\overrightarrow{OA} \times \ \overrightarrow{OB} = -12\vec{p} \times \vec{q} - 2\vec{p} \times \vec{q} = -14\vec{p} \times \vec{q}$$

Thus, the area of the parallelogram is

$$S = \left|\overrightarrow{OA} \times \ \overrightarrow{OB}\right| = |-14\vec{p} \times \vec{q}| = |-14| \cdot |\vec{p} \times \vec{q}| = 14\ |\vec{p} \times \vec{q}|$$

Using the formula of the magnitude of a cross product:

$$|\vec{p} \times \vec{q}| = |\vec{p}|\ |\vec{q}|\ \sin\angle(\vec{p};\vec{q}) = 2 \cdot 1 \cdot \sin\frac{\pi}{3} = 2 \cdot 1 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$

we obtain

$$S = \left|\overrightarrow{OA} \times \ \overrightarrow{OB}\right| = 14\ |\vec{p} \times \vec{q}| = 14\sqrt{3}$$

Therefore, the area of the parallelogram is $S = 14\sqrt{3}$.

2) Let us find the lengths of the diagonals of the parallelogram, OD and AB.

The lengths of the diagonals OD and AB are equal to the magnitudes of the vectors $\overrightarrow{\mathrm{OD}}$ and $\overrightarrow{AB}$, which can be expressed in terms of the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$:

$$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{OB}, \quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

Substituting the given expressions for vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$, we obtain:

$$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{OB} = (3\vec{p} + 2\vec{q}) + (\vec{p} - 4\vec{q}) = 4\vec{p} - 2\vec{q}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (\vec{p} - 4\vec{q}) - (3\vec{p} + 2\vec{q}) = -2\vec{p} - 6\vec{q}$$

Since the coordinates of the vectors are not known, we use the property of the dot product:

$$|\vec{a}|^2 = \vec{a}^2$$

First, find the square of the magnitude of the vector $\overrightarrow{OD}$ :

$$\left|\overrightarrow{OD}\right|^2 = \overrightarrow{OD}^2 = (4\vec{p} - 2\vec{q})^2 = 16\vec{p}^2 - 16\vec{p} \cdot \vec{q} + 4\vec{q}^2$$

Using properties of the dot product and the formula

$$\vec{p} \cdot \vec{q} = |\vec{p}|\,|\vec{q}| cos\,\angle(\vec{p}, \vec{q}) = 2 \cdot 1 \cdot \cos\frac{\pi}{3} = 2 \cdot 1 \cdot \frac{1}{2} = 1,$$

we obtain:

$$\left|\overrightarrow{OD}\right|^2 = 16|\vec{p}|^2 - 16 \cdot 1 + 4|\vec{q}|^2 = 16 \cdot 2^2 - 16 + 4 \cdot 1^2 = 52$$

Therefore,

$$\left|\overrightarrow{OD}\right| = \sqrt{52} = 2\sqrt{13}$$

Similarly, we find the square of the magnitude of the vector $\overrightarrow{AB}$ :

$$\left|\overrightarrow{AB}\right|^2 = \overrightarrow{AB}^2 = (2\vec{p} + 6\vec{q})^2 = 4\vec{p}^2 + 24\vec{p}\vec{q} + 36\vec{q}^2 = 4|\vec{p}|^2 + 24 \cdot 1 + 36|\vec{q}|^2 =$$

$$= 4 \cdot 2^2 + 24 + 36 \cdot 1^2 = 76$$

Hence,

$$\left|\overrightarrow{AB}\right| = \sqrt{76} = 2\sqrt{16}$$

The area of the parallelogram is $S = 14\sqrt{3}$.

The lengths of its diagonals are $OD = \sqrt{52} = 2\sqrt{13}$ and $AB = 2\sqrt{16}.$

***Example 5****.*

A force $\vec{\mathrm{F}} = 5\vec{\imath} - 2\vec{\jmath} + 4\vec{k}$ is applied At point M(1,2,3). Find the moment of this force about point A(3,2,-1).

Solution:

The moment of a force $\vec{\mathrm{F}}$ about the point A is given by

$$\vec{M} = \overrightarrow{AM} \times \vec{F}.$$

We find vector $\overrightarrow{AM}$

$$\overrightarrow{AM} = (1 - 3, 2 - 2, 3 - (-1)) = (-2, 0, 4).$$

Then,

$$\vec{M} = \overrightarrow{AM} \times \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -2 & 0 & 4 \\ 5 & -2 & 4 \end{vmatrix} = \begin{vmatrix} 0 & 4 \\ -2 & 4 \end{vmatrix} \vec{\imath} - \begin{vmatrix} -2 & 4 \\ 5 & 4 \end{vmatrix} \vec{\jmath} + \begin{vmatrix} -2 & 0 \\ 5 & -2 \end{vmatrix} \vec{k} = 8\vec{\imath} + 28\vec{\jmath} + 4\vec{k}$$

Thus, the moment of a force $\vec{F}$ about point A is

$$\vec{M} = 8\vec{\imath} + 28\vec{\jmath} + 4\vec{k}$$

### 2.5.3. Scalar Triple Product (Mixed Products)

#### 2.5.3.1 Scalar Triple Product: Definition, Geometric Interpretation, and Properties

Definition: The scalar triple product is defined as the dot product of one of the vectors with the cross product of the other two:

$$(\vec{a} \times \vec{b}) \cdot \vec{c}$$

The scalar triple product is also called the mixed product or box product. The triple scalar product is a scalar (number).

Geometric interpretation of the scalar triple product

Let us determine the geometric meaning of the expression $(\vec{a} \times \vec{b}) \cdot \vec{c}$.

Let us construct a parallelepiped whose edges are the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, drawn from the same point.
Also, construct the vector $\vec{l} = \vec{a} \times \vec{b}$, which is perpendicular to vectors $\vec{a}$ and $\vec{b}$. (Fig.44).

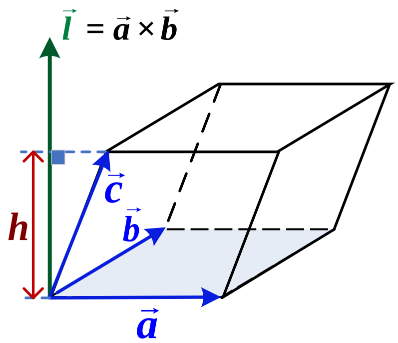



*Figure 44*. Parallelepiped formed by vectors $\vec{a}$, $\vec{b}$,and $\vec{c}$.

Then,

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{l} \cdot \vec{c} = |\vec{l}| \cdot \mathrm{proj}_{\vec{l}}\vec{c}.$$

By the properties of the dot product, the magnitude of vector $\vec{l}$ is

$$|\vec{l}| = |\vec{a} \times \vec{b}| = S_{parallelogram}$$

where $S_{parallelogram}$ is the area of the parallelogram, constructed on vectors $\vec{a}$ and $\vec{b}$.

The projection of $\vec{c}$ onto $\vec{l}$ is equal to

$$proj_{\vec{l}}\vec{c} = \pm h,$$

where $h$ is the height of the parallelepiped.

The sign is determined by the orientation of the triple $\vec{a}, \vec{b}, \vec{c}$:

'+' sign, if $\vec{a}, \vec{b}, \vec{c}$ form the right-handed triple (in this case $\vec{c} \uparrow\uparrow \vec{l}$)

'- ' sign, if $\vec{a}, \vec{b}, \vec{c}$ form the left-handed triple (then $\vec{c} \uparrow\downarrow \vec{l}$ )

Thus,

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = |\vec{l}| \cdot proj_{\vec{l}}\vec{c} = S_{parallelogram} \cdot (\pm h) = \pm V,$$

where $V$ is the volume of the parallelepiped formed by the vectors $\vec{a}, \vec{b}, \vec{c}$.

Therefore, the scalar triple product of three vectors $(\vec{a} \times \vec{b}) \cdot \vec{c}$ equal to the volume $V$ of the parallelepiped constructed on these vectors, taken with a plus sign if the vectors form a right-handed triple, and with a minus sign if they form a left-handed triple.

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{cases} V, & \text{if } \vec{a}, \vec{b}, \vec{c} \text{ form right} - \text{handed triple} \\ -V, & \text{if } \vec{a}, \vec{b}, \vec{c} \text{ form left} - \text{handed triple} \end{cases}$$

**Properties of the scalar triple product**

1. The scalar triple product does not change when the vector and scalar products swap positions:

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c})$$

***Proof.*** According to the geometric interpretation of the scalar triple product:

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \pm V \quad \text{and} \quad \vec{a} \cdot (\vec{b} \times \vec{c}) = \pm V$$

We take the same sign for both equalities, because the triples of vectors $\vec{a}, \vec{b}, \vec{c}$ have the same orientation.

Therefore,

$$\left(\vec{a} \times \vec{b}\right) \cdot \vec{c} = \vec{a} \cdot \left(\vec{b} \times \vec{c}\right).$$

Q.E.D.

Taking into account this property, often the scalar triple product is written without symbols of vector or scalar multiplication:

$$\left(\vec{a} \times \vec{b}\right) \cdot \vec{c} = \vec{a}\vec{b}\vec{c}$$

2. The scalar triple product does not change under a cyclic permutation of its factors:

$$\left(\vec{a} \times \vec{b}\right) \cdot \vec{c} = \left(\vec{b} \times \vec{c}\right) \cdot \vec{a} = (\vec{c} \times \vec{a}) \cdot \vec{b}$$

It happens because neither the volume of the parallelepiped nor the orientation of its edges changes.

3. The scalar triple product changes its sign when any two of its factors are interchanged:

$$\vec{a}\vec{b}\vec{c} = -\vec{a}\vec{c}\vec{b}, \qquad \vec{a}\vec{b}\vec{c} = -\vec{b}\vec{a}\vec{c}, \qquad \vec{a}\vec{b}\vec{c} = -\vec{c}\,\vec{b}\vec{a}$$

It happened because such a permutation of the factors in the vector product changes the orientation of vector triples.

4. The scalar triple product of nonzero vectors $\vec{a}, \vec{b}, \vec{c}$ is equal to zero if and only if they are coplanar.

$$\vec{a}\vec{b}\vec{c} = 0 \quad \Leftrightarrow \quad \vec{a}, \vec{b}, \vec{c} \text{ are coplanar}$$

***Proof***. A) Let $\vec{a}\vec{b}\vec{c} = 0$.

Assume that the vectors are not coplanar. Then it is possible to construct a parallelepiped on these vectors with volume $V \neq 0$ which implies that $\vec{a}\vec{b}\vec{c} \neq 0$. It contradicts the assumption $\vec{a}\vec{b}\vec{c} = 0$ .

B) Now let the vectors $\vec{a}, \vec{b}, \vec{c}$ be coplanar (i.e., lie in one plane or in parallel planes).

In this case, the vectors $\vec{a} \times \vec{b}$ and $\vec{c}$ are perpendicular and thus $\left(\vec{a} \times \vec{b}\right) \cdot \vec{c} = 0$, by the property of the scalar product. Q.E.D.

### 2.5.3.2. Mixed Product in Terms of Coordinates

Let the vectors $\vec{a}, \vec{b}, \vec{c}$ are given:

$$\vec{a} = a_x\vec{\imath} + a_y\vec{\jmath} + a_z\vec{k}\,, \quad \vec{b} = b_x\vec{\imath} + b_y\vec{\jmath} + b_z\vec{k}, \quad \vec{c} = c_x\vec{\imath} + c_y\vec{\jmath} + c_z\vec{k}$$

The mixed product using its definition and properties of cross and dot products:

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \cdot \vec{c} =$$

$$= \left( \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} \vec{\imath} - \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} \vec{\jmath} + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \vec{k} \right) \cdot (c_x\vec{\imath} + c_y\vec{\jmath} + c_z\vec{k}) =$$

$$= \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} c_x - \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} c_y + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} c_z$$

The resulting expression corresponds to the expansion of a third-order determinant along its 3rd row. Hence, it can be rewritten in the following compact form:

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

### 2.5.3.3. Some Applications of the Mixed Product

1. *Checking the coplanarity of vectors*

   $$\vec{a}\vec{b}\vec{c} = 0 \quad \Leftrightarrow \quad \vec{a}, \vec{b}, \vec{c} \text{ are coplanar}$$

2. *Checking the orientation of a triple of vectors:*

   If $\vec{a}\vec{b}\vec{c} > 0$ , then vectors $\vec{a}, \vec{b}, \vec{c}$ form a right-handed triple.

   If $\vec{a}\vec{b}\vec{c} < 0$ , then vectors $\vec{a}, \vec{b}, \vec{c}$ form a left-handed triple.

3. *Definition of the volumes of a parallelepiped and a triangular pyramid:*

   The volume of a parallelepiped constructed on non-coplanar vectors $\vec{a}, \vec{b}, \vec{c}$ as its edges (Fig.45), is equal to the absolute value of the mixed product of these vectors

   $$V = |(\vec{a} \times \vec{b}) \cdot \vec{c}|$$

The volume of a triangular pyramid constructed on the vectors $\vec{a}, \vec{b}, \vec{c}$ (Fig.46) is exactly one-sixth of the volume of the parallelepiped constructed on these vectors. Therefore, the volume of a triangular pyramid is

$$V = \frac{1}{6}\left|(\vec{a} \times \vec{b}) \cdot \vec{c}\right|.$$

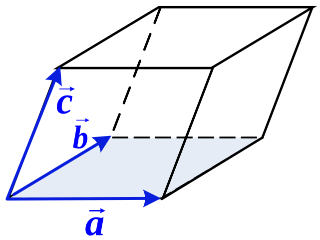



*Figure 45*. Parallelepiped formed by vectors $\vec{a}$, $\vec{b}$,and $\vec{c}$.

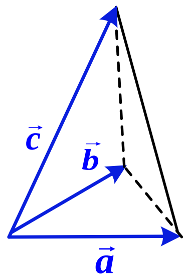



*Figure 46.* Triangular pyramid formed by vectors $\vec{a}$, $\vec{b}$,and $\vec{c}$.

### 2.5.3.4. Examples of Solved Problems

***Example 1.***

Determine the orientation of the triple of vectors $\vec{a} = \vec{\imath}$, $\vec{b} = -\vec{\imath} - 2\vec{\jmath}$, and $\vec{c} = \vec{\imath} + 3\vec{k}$. Is it right-handed or left-handed**?**

Solution

The vectors in coordinate form:

$$\vec{a} = (1,0,0), \quad \vec{b} = (-1,-2,0), \quad \vec{c} = (1,0,3).$$

To determine the orientation of the triple of vectors, we compute their mixed product

$$\vec{a}\vec{b}\vec{c} = (\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{vmatrix} 1 & 0 & 0 \\ -1 & -2 & 0 \\ 1 & 0 & 3 \end{vmatrix} = -6$$

Thus,

$$\vec{a}\vec{b}\vec{c} = -6 < 0.$$

Since the mixed product is negative, the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ form a left-handed triple.

***Example 2.***

A parallelepiped is constructed on the vectors $\overrightarrow{OA} = -2\vec{\imath} + \vec{\jmath}$, $\overrightarrow{OB} = \vec{\imath} + 2\vec{\jmath} - \vec{k}$, $\overrightarrow{OC} = \vec{\imath} + \vec{\jmath} + 3\vec{k}$ . Find its volume.

Solution:

The volume of the parallelepiped constructed on the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ is equal to the modulus of its mixed product

$$V = \left|\left(\overrightarrow{\mathrm{OA}} \times \overrightarrow{OB}\right) \cdot \overrightarrow{OC}\right|$$

$$\left(\overrightarrow{\mathrm{OA}} \times \overrightarrow{OB}\right) \cdot \overrightarrow{OC} = \begin{vmatrix} -2 & 1 & 0 \\ -1 & 2 & -1 \\ 1 & 1 & 3 \end{vmatrix} = -12 + 0 + 1 - 0 - 2 + 3 = -10$$

$$V = \left|\left(\overrightarrow{\mathrm{OA}} \times \overrightarrow{OB}\right) \cdot \overrightarrow{OC}\right| = |-10| = 10$$

***Example 3.***

Points A(2,-1,0), B(0,3,-1), C(-1;2,1) and D(1,3,4) are given. Find the volume of the triangular pyramid ABCD and the height drawn from the vertex D (Fig.37).

Solution:

1. The volume of the pyramid

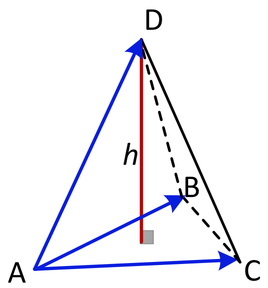



Figure 47. Triangular pyramid ABCD from Example 3

The triangular pyramid ABCD is based on three vectors originating from one vertex, for example, vertex A: $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$.

Let us find these vectors:

$$\overrightarrow{AB} = (0 - 2, 3 - (-1), -1 - 0) = (-2, 4, -1)$$

$$\overrightarrow{AC} = (-1-2, 2-\ (-1), 1-\ 0) = (-3, 3, 1)$$

$$\overrightarrow{AD} = (1-2, 3-\ (-1), 4-\ 0) = (-1\ ,4,4)$$

The volume of a triangular pyramid based on vectors $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ is equal to one-sixth of the absolute value of their scalar triple product:

$$V = \frac{1}{6}\left|\left(\overrightarrow{AB} \times \overrightarrow{AC}\right) \cdot \overrightarrow{AD}\right|$$

Let us find the triple scalar product of the vectors $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$:

$$\left(\overrightarrow{AB} \times \overrightarrow{AC}\right) \cdot \overrightarrow{AD} = \begin{vmatrix} -2 & 4 & -1 \\ -3 & 3 & 1 \\ -1 & 4 & 4 \end{vmatrix} = -24 + 12\ -\ 4 - 3 + 8 + 48 = 37$$

As a result, the volume of pyramid ABCD is

$$V = \frac{1}{6}|37| = \frac{37}{6} \approx 6.2\ \ (cubic\ units)$$

2. The height drawn from vertex D

The volume of a triangular pyramid *ABCD* equals one third of the area of the base multiplied by the height *h*:

$$V = \frac{1}{3}h \cdot S_{base}$$

If we consider the height *h* drawn from vertex D to the base *ABC*, then it follows from this formula that

$$h = \frac{3V}{S_{ABC}}$$

The area $S_{ABC}$ of the triangle *ABC*, constructed on vectors $\overrightarrow{AB}, \overrightarrow{AC}$ is equal to one half of the magnitude of their cross product:

$$S_{ABC} = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$$

Let us find the cross product of vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -2 & 4 & -1 \\ -3 & 3 & 1 \end{vmatrix} = \vec{\imath}\begin{vmatrix} 4 & -1 \\ 3 & 1 \end{vmatrix} - \vec{\jmath}\begin{vmatrix} -2 & -1 \\ -3 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} -2 & 4 \\ -3 & 3 \end{vmatrix} = 7\vec{\imath} + 5\vec{\jmath} + 6\vec{k}$$

Then,

$$S_{ABC} = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\sqrt{7^2 + 5^2 + 6^2} = \frac{\sqrt{110}}{2}$$

Therefore, the height drawn from vertex D to the base *ABC* is equal:

$$h = \frac{3V}{S_{ABC}} = 3 \cdot \frac{37}{6} \cdot \frac{2}{\sqrt{110}} = \frac{37}{\sqrt{110}} = \frac{37\sqrt{110}}{110} \approx 3.5$$

***Example 4***.

Check whether the vectors $\vec{a} = 2\vec{\imath} + 3\vec{\jmath} + \vec{k}$ , $\vec{b} = -\vec{\imath} - \vec{k}$, $\vec{c} = 2\vec{\imath} + 2\vec{\jmath} + 2\vec{k}$ are coplanar.

Solution.

If three vectors are coplanar, then their scalar triple product is equal to zero **.**

Let us find the scalar triple product of vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$:

$$\vec{a}\vec{b}\vec{c} = \begin{vmatrix} 2 & 3 & 1 \\ -1 & 0 & -1 \\ 2 & 2 & 2 \end{vmatrix} = 0 - 2 - 6 - 0 + 4 + 6 = 2 \neq 0$$

Since $\vec{a}\vec{b}\vec{c} \neq 0$, vectors $\vec{a}, \vec{b}, \vec{c}$ are not coplanar.

***Example 5***.

Check whether the points *A(0,1,6), B(-2,0,1), C(1,-1,1),* and *D(-1,2,8)* lie in the same plane.

Solution

Let us consider three vectors originating from the same point, for example, from point B:

$$\overrightarrow{BA} = (0 - (-2), 1 - 0, 6 - 1) = (2, 1, 5)$$

$$\overrightarrow{BC} = (1 - (-2), -1 - 0, 1 - 1) = (3, -1, 0)$$

$$\overrightarrow{BD} = (-1 - (-2), 2 - 0, 8 - 1) = (1, 2, 7)$$

If all four points *A, B, C, D* lie in the same plane, then the vectors $\overrightarrow{BA}$, $\overrightarrow{BC}$, and $\overrightarrow{BD}$ are coplanar (Fig.48).

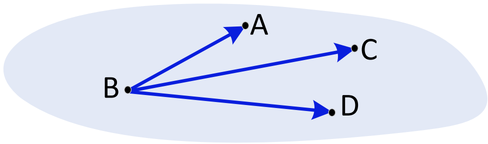



Figure 48. Points A, B, C, and D lie in the same plane.

To check whether these vectors are coplanar, we compute their scalar triple product:

$$\left(\overrightarrow{BA} \times \overrightarrow{BC}\right) \cdot \overrightarrow{BD} = \begin{vmatrix} 2 & 1 & 5 \\ 3 & -1 & 0 \\ 1 & 2 & 7 \end{vmatrix} = -14 + 30 + 0 - (-5) - 0 - 21 = 0$$

Since

$$\left(\overrightarrow{BA} \times \overrightarrow{BC}\right) \cdot \overrightarrow{BD} = 0,$$

the vectors $\overrightarrow{BA}$, $\overrightarrow{BC}$, and $\overrightarrow{BD}$ are coplanar. Therefore, the points *A, B, C,* and *D* lie in one plane.

### 2.6. Examples of Vector Applications

Vectors are a powerful and indispensable mathematical tool for the analysis of physical processes and engineering systems, allowing the description of quantities that possess both magnitude and direction. Vector analysis is widely applied in physics, engineering, robotics, navigation, GPS navigation, computer technology, and other fields of science and technology.

Fundamental quantities such as force, velocity, acceleration, and displacement are described using vectors.

Let us consider several examples of the application of vectors in different areas of science and technology.

**In mechanics, mechanical engineering,** and robotics, vectors are used to analyse forces acting on objects, calculate structural loads (e.g., on bridges, buildings, and machine components), and study the motion of rigid bodies and the kinematics of mechanisms. They form the mathematical foundation of statics, kinematics, and dynamics, and are essential in stress analysis, vibration theory, and the modelling of dynamic systems.

Vector operations such as the scalar (dot) product and the vector (cross) product play a fundamental role in these applications. The scalar product is used, for example, to determine the work done by a force. The vector (cross) product is used to describe rotational effects, such as moment of force (torque) and the linear velocity of a point in a rotating rigid body.

Problem 1.

A medium-class 6-axis industrial robot holds a payload of mass $m = 10$ kg. A fixed Cartesian coordinate system is attached to the robot base and is shown in Fig.49.

The center of the robot's elbow joint is located at point $B(0.2, 0.25, 0.7)\ m$.

The tool flange (end-effector mounting point) is located at point $E(0.5, 0.6, 0.9)m$.

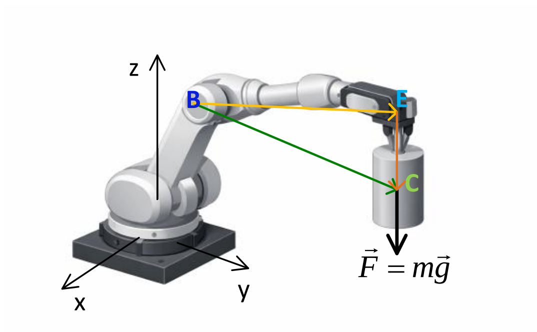


*Figure 49*. An industrial robot holds a payload

The payload is rigidly grasped. Its center of mass C is located $d = 0.12$m below the tool flange along the negative Z-axis.

Find the moment of the force (torque) of the payload weight about the elbow joint point B and its magnitude.

*Assumptions*: Static analysis. Only the payload weight is considered; the self-weight of the robot links (arm/forearm) and the gripper is neglected.

Solution

The magnitude of the gravitational force acting on the payload is

$$mg = 10 \cdot 9.81 \ = 98.1 \text{ N},$$

Where

$g$ is the gravitational acceleration $g = 9.81\ m/s^2$.

The weight force vector applied at point **C** is

$$\vec{F} = (0, 0, -98.1)\ N.$$

Find the position of the center of mass C. Since the center of mass is located 0.12 m below the flange along the negative Z-axis,

$$C\ (0.5, 0.6, 0.9 - 0.12)$$

$$C\ (0.5, 0.6, 0.78)$$

Find the radius vector from the robot's elbow joint center B to the payload center of mass C

$$\vec{r}_{BC} = \overrightarrow{BC} = (0.5 - 0.2, 0.6 - 0.25, 0.78 - \ 0.7) = (0.3, 0.35, 0.08)\ m$$

The force moment about point B is given by the cross product

$$\vec{M}_B = \vec{r}_{BC} \times \vec{F}$$

$$\vec{M}_B = (0.3, 0.35, 0.08) \times (0, 0, -98.1) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.3 & 0.35 & 0.08 \\ 0 & 0 & -98.1 \end{vmatrix} =$$

$$= \vec{i}\begin{vmatrix} 0.35 & 0.08 \\ 0 & -98.1 \end{vmatrix} - \vec{j}\begin{vmatrix} 0.3 & 0.08 \\ 0 & -98.1 \end{vmatrix} + \vec{k}\begin{vmatrix} 0.3 & 0.35 \\ 0 & 0 \end{vmatrix}.$$

$$\vec{M}_B = -34.34\,\vec{i} + 29.43\vec{j}$$

The magnitude of the force momentum acted on the B is

$$\left|\vec{M}_B\right| = \sqrt{(-34.34)^2 + (29.43)^2 + 0^2} = 45.2$$

*Answer:* $\vec{M}_B = -34.34\,\vec{i} + 29.43\vec{j}$, $\left|\vec{M}_B\right| = 45.2$

**Note.** It is important to calculate the torque to determine how much torque the elbow motor must generate to hold the payload in this position.

**In electrodynamics and electrical engineering**, vectors are essential for describing electric and magnetic fields, electromagnetic forces acting on charged particles, and analysing phenomena in alternating current (AC) circuits. Vector analysis enables the modelling and calculation of complex electromagnetic processes.

Electric and magnetic fields are described by the electric field intensity vector $\vec{E}$ and the magnetic flux density vector $\vec{B}$. The vector (cross) product is used to describe electromagnetic forces acting on conductors or charged particles.

Problem 2.

A straight conductor segment of length 0.2 m carries a current of 5 A. It is placed in a uniform magnetic field with a magnetic flux density vector $\vec{B} = (0.3, 0.1, 0.2)$ T. The conductor is oriented along the vector $\vec{l} = (2,0,1)$ . Find the force acting on the current-carrying conductor.

*Solution*

A straight current-carrying conductor placed in a uniform magnetic field experiences the Ampère force, which is directed perpendicular to both the current and the magnetic field and is given by the formula:

$$\vec{F}_{\mathrm{A}} = I\left(\vec{L} \times \vec{B}\right)$$

where
$\vec{B}$ is the magnetic flux density vector,
$L$ is the length of the current-carrying conductor,
$I$ is the electric current.
$\vec{L} = L \cdot \vec{e}$ is the length vector of the conductor (directed along the current).
$\vec{e}$ is a unit vector along the current-carrying conductor

First, we find the unit vector $\vec{e}$ along the conductor by normalizing the direction vector $\vec{l} = (2, 0, 1)$

$$|\vec{l}| = \sqrt{2^2 + 0^2 + 1^2} = \sqrt{5}$$

The unit direction vector is

$$\vec{e} = \frac{\vec{l}}{|\vec{l}|} = \left(\frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}\right)$$

The length vector of the conductor

$$\vec{L} = L\vec{e} = 0.2\ \left(\frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}\right) = \left(\frac{0.4}{\sqrt{5}}, 0, \frac{0.2}{\sqrt{5}}\right)$$

Find the cross product:

$$\vec{L} \times \vec{B} = \left(\frac{0.4}{\sqrt{5}}, 0, \frac{0.2}{\sqrt{5}}\right) \times (0.3, 0.1, 0.2) = \frac{0.2}{\sqrt{5}} \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2 & 0 & 1 \\ 0.3 & 0.1 & 0.2 \end{vmatrix} =$$

$$= \frac{0.2}{\sqrt{5}}\left(\vec{\imath} \begin{vmatrix} 0 & 1 \\ 0.1 & 0.2 \end{vmatrix} - \vec{\jmath} \begin{vmatrix} 2 & 1 \\ 0.3 & 0.2 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & 0 \\ 0.3 & 0.1 \end{vmatrix}\right) =$$

$$= \frac{0.2}{\sqrt{5}}\left(-0.1\,\vec{\imath} + 0.1\vec{\jmath} - 0.2\vec{k}\right)$$

Therefore, the Ampère force acting on the conductor segment is:

$$\vec{F}_{\mathrm{A}} = I\left(\vec{L} \times \vec{B}\right) = 5 \cdot \frac{0.2}{\sqrt{5}}\left(-0.1\,\vec{\imath} + 0.1\vec{\jmath} - 0.2\vec{k}\right) = -\frac{0.1}{\sqrt{5}}\vec{\imath} + \frac{0.1}{\sqrt{5}}\vec{\jmath} - \frac{0.2}{\sqrt{5}}\vec{k}$$

$$\vec{F}_{\mathrm{A}} \approx\ (-0.0447, 0.0447, -0.0894)\ N$$

The magnitude of the force is

$$|\vec{F}_{\mathrm{A}}| = \sqrt{\left(-\frac{0.1}{\sqrt{5}}\right)^2 + \left(\frac{0.1}{\sqrt{5}}\right)^2 + \left(-\frac{0.2}{\sqrt{5}}\right)^2} = 0.1\ N$$

*Answer:* The Ampère force acting on the conductor segment is $\vec{F}_{\mathrm{A}} \approx (-0.0447,\ 0.0447,\ -0.0894)\ N$**;** The magnitude of the force is $F = |\vec{F}_{\mathrm{A}}| = 0.1\ N$.

**Note:** If a closed current-carrying loop is placed in a uniform magnetic field, the Ampère forces acting on opposite sides of the loop form a force couple that produces a torque. This torque causes the loop to rotate until its magnetic moment becomes aligned with the magnetic field. This principle is used, for example, in **electric motor**s and in **electrical measuring instruments**.

Let us consider the other problem of an electromagnetic force acting on a moving charge. A charged particle with charge $q$, moving with velocity $\vec{v}$ in a uniform magnetic field $\vec{B}$, experiences the Lorentz force, which is perpendicular to both the velocity and the magnetic field, and is given by the formula

$$\vec{\mathrm{F}}_L = q(\vec{v} \times \vec{B}).$$

If a charged particle moves in a region where both electric $\vec{E}$ and magnetic fields are present, the total Lorentz force is

$$\vec{\mathrm{F}}_L = q(\vec{E} + \vec{v} \times \vec{B}).$$

The Lorentz force causes a charged particle to move in a circular path or along a helical trajectory (Fig.50(b)). If the particle's velocity is perpendicular to the magnetic field, it moves in a circle, with a radius that depends on its speed and the magnetic field strength (Fig.50(a)). If there is a component of velocity parallel to the magnetic field, the particle follows a helical path.

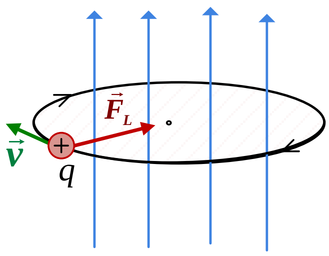



*Figure 50 (a)*. Circular path

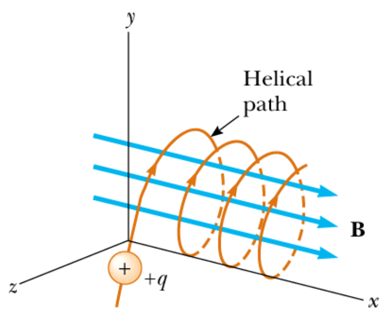



*Figure 50 (b)*. Helical path *(Picture is taken from https://wedophysics.blogspot.com)*

This phenomenon underlies the operation of cathode-ray tubes (CRTs), mass spectrometers, particle accelerators, magnetrons (used in microwave ovens), and electron microscopes for beam focusing. It also plays an important role in plasma physics and astrophysics.

- In mass spectrometers, a magnetic field forces ions to move along circular paths whose radii depend on their mass-to-charge ratio, enabling particle separation.
- In particle accelerators such as cyclotrons and synchrotrons, widely used in fundamental research and modern medical applications, magnetic fields confine charged particles to circular or spiral paths, while electric fields provide the energy necessary for their acceleration.
- Magnetrons generate microwave radiation due to the motion of electrons in crossed electric and magnetic fields and are used in microwave ovens and radar systems.

Problem 3.

An electron with a charge $\mathrm{q} = -1.60 \cdot 10^{-19}\ \mathrm{C}$ moves in a cathode-ray tube with velocity $\vec{v} = (2.0 \cdot 10^{6}, 0, 0)\ \mathrm{m/s}$ in a magnetic field that is the superposition of two fields: the magnetic field of the coils $\vec{\mathrm{B}}_{\mathrm{coil}} = (0, 0, 2.0 \cdot 10^{-3})\ \mathrm{T}$ and the Earth's magnetic field (horizontal component, typically tens of μT) $\vec{\mathrm{B}}_{\mathrm{e}} = (0, 3.0 \cdot 10^{-5}, 0)\ \mathrm{T}$

The coordinate axes are chosen as follows:

- the z-axis is along the axis of the coils,
- the y-axis points north (horizontally),
- the x-axis completes the right-handed coordinate system.

Find the Lorentz force acting on the electron.

*Solution*

The resulting magnetic field is

$$\vec{\mathrm{B}} = \vec{\mathrm{B}}_{\mathrm{coil}} + \vec{\mathrm{B}}_{e} = (0, 0, 2.0 \cdot 10^{-3}) + (0, 3.0 \cdot 10^{-5}, 0) = (0, 3.0 \cdot 10^{-5}, 2.0 \cdot 10^{-3})$$

The Lorentz force in a magnetic field is given by

$$\vec{\mathrm{F}}_{L} = q(\vec{v} \times \vec{B}).$$

We compute the cross product:

$$\vec{v} \times \vec{B} = (2.0 \cdot 10^{6}, 0, 0) \times (0, 3.0 \cdot 10^{-5}, 2.0 \cdot 10^{-3})$$

$$= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2 \cdot 10^6 & 0 & 0 \\ 0 & 3 \cdot 10^{-5} & 2 \cdot 10^{-3} \end{vmatrix} = -4 \cdot 10^3 \, \vec{\jmath} \, + 60 \, \vec{k}$$

The Lorentz force acting on the electron

$$\vec{\mathrm{F}}_L = q(\vec{v} \times \vec{B}) = -1.60 \cdot 10^{-19} \cdot (-4 \cdot 10^3 \, \vec{\jmath} \, + 60 \, \vec{k}) = 6.4 \cdot 10^{-16} \vec{\jmath} - \, 9.6 \cdot 10^{-18} \, \vec{k}$$

The magnitude of the Lorenc force

$$|\vec{\mathrm{F}}_L| = \sqrt{0^2 + (6.4 \cdot 10^{-16})^2 + (-\, 9.6 \cdot 10^{-18})^2} \approx 6.4 \cdot 10^{-16} \quad N.$$

*Answer*

The Lorentz force acting on the electron is $\vec{\mathrm{F}}_L = 6.4 \cdot 10^{-16} \vec{\jmath} - \, 9.6 \cdot 10^{-18} \, \vec{k}$ N ,
the magnitude of the force is $\mathrm{F} = |\vec{\mathrm{F}}_L| \approx 6.4 \cdot 10^{-16} \quad N.$

**Vectors in Navigation**:

In navigation problems, one often needs to determine the actual (ground) speed and the true course of a ship or an aircraft relative to the Earth, taking into account the influence of wind and/or water current. Such problems are formulated using vector algebra. Each velocity is represented by a vector whose direction corresponds to the direction of motion of the vehicle or to the direction of the wind or current, while its speed is interpreted as the magnitude of the corresponding vector. The velocity relative to the Earth is obtained as the vector sum of the vehicle's velocity relative to the medium and the velocity of the medium itself.

In most air and marine navigation problems, a two-dimensional Cartesian coordinate system is introduced in the horizontal plane in the following way:

OX axis is directed toward geographic (true) north, OY axis is directed toward east.

This choice is convenient because aviation and marine bearings are measured clockwise from north (0°–360°). In this case:

bearing 0° - means traveling due North;
bearing 90° - means traveling due East
bearing 180° - means traveling due South
bearing 270° - means traveling due West.

The direction of motion may also be expressed using cardinal directions. In this notation, the direction is specified as an angular deviation from one of the principal directions (north, south, east, or west).

For example, if a boat travels 30° from north toward east, the direction may be expressed as 30° east of north (N30E), which is equivalent to a bearing of 030°. Similarly, if a boat travels 60° from south toward east, the direction may be expressed as 60° east of south (S60E), which corresponds to a bearing of 120°.

Problem 4

The aircraft is flying with an airspeed 650 km/h in the direction of North.
The wind is blowing in the direction $60^0$ East of South with a velocity 90 km/h

Find the aircraft's actual (ground) velocity and the real course of the aircraft over the ground (the angle it makes with the north direction).

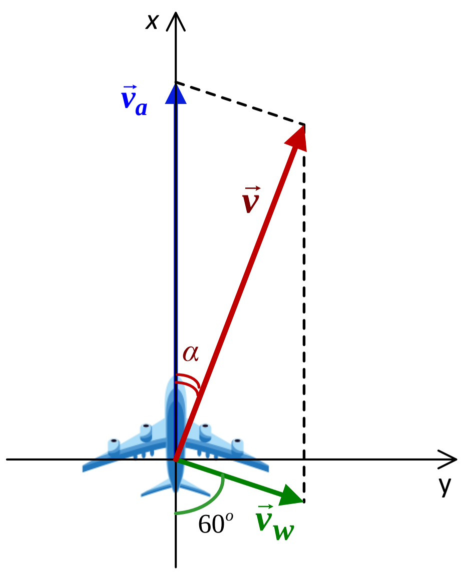



*Figure 51*. Illustration of Problem 4.

Solution:

1) We introduce the airspeed velocity vector $\overrightarrow{v_\mathrm{a}}$ and the velocity vector of the wind $\overrightarrow{v_\mathrm{w}}$.

The airspeed velocity of the aircraft is 650 km/h, and the angle with the direction of North is 0°, which means that

$$\overrightarrow{v_a} = (650{,}0) \text{ km/h}$$

The wind is blowing at a velocity 90 km/h, and the clockwise angle between the wind velocity vector and the direction of North is $180^0 - 60^0 = 120^0$ , It means that the magnitude of the wind velocity vector is $|\overrightarrow{v_w}| = 90$ and the components of the vector are

$$v_{w_x} = |v_w|\, cos120^0 = 90 \cdot \left(-\frac{1}{2}\right) = -45$$

$$v_{w_y} = |v_w|\, sin120^0 \ = 90 \cdot \frac{\sqrt{3}}{2} = 45\sqrt{3}$$

Hence, the velocity vector of the wind is

$$\overrightarrow{v_w} = (-45,\ \ 45\sqrt{3})\ \text{km/h}$$

2) The resultant velocity is the vector sum:

$$\vec{v} = \overrightarrow{v_a} + \overrightarrow{v_w} = (650, 0) + (-45,\ \ 45\sqrt{3}) = (605, 45\sqrt{3})\ \ \mathrm{km/h}$$

The aircraft's actual (ground) velocity is equal to the magnitude of the resultant velocity:

$$|\ \vec{v}\ | = \sqrt{605^2 + \left(45\sqrt{3}\right)^2} = \sqrt{366025 + 6075} \approx \sqrt{372100} \quad \approx 610\ \text{km/h}$$

3) The actual course of the aircraft or the course over ground is the angle of resultant velocity with the North direction (The angle with the $OX$-axis)

$$\cos\alpha = \frac{v_x}{|\ \vec{v}\ |} = \frac{605}{610} \approx 0.9918 \Rightarrow \alpha = \arccos(0.9918) \approx 7.34°$$

It means the actual course of the aircraft is $7.34°$ east of North or the aircraft is on the bearing $7.34°$over ground.

Note that the actual velocity and the angle α can also be determined from the so-called wind (or current) triangle using the law of sines and the law of cosines.

*Answer:* The true speed of the aircraft is approximately 610 km/h, and the deviation from the north direction is approximately 7.34°.

Consider another classical type of navigation problem.

In practical navigation, a vessel or an aircraft often travels along several successive legs and changes direction during the journey. In such problems, it is required to determine the resultant displacement and the bearing (direction) from the starting point.

Each leg of the route is represented by a displacement vector whose magnitude equals the distance travelled and whose direction corresponds to the bearing. The overall displacement and the final direction are then obtained by vector addition of these segment vectors.

Problem 5

A cargo ship leaves a port and sails 48 nautical miles on a bearing of 315°. It then changes course and sails 60 miles on a bearing of 250° (Fig.52).

Determine:

1) the resultant displacement vector of the ship

2) the distance from the port to the ship

3) The bearing of the final position of the ship from the port. (the angle between the resultant displacement vector and OX-axis)

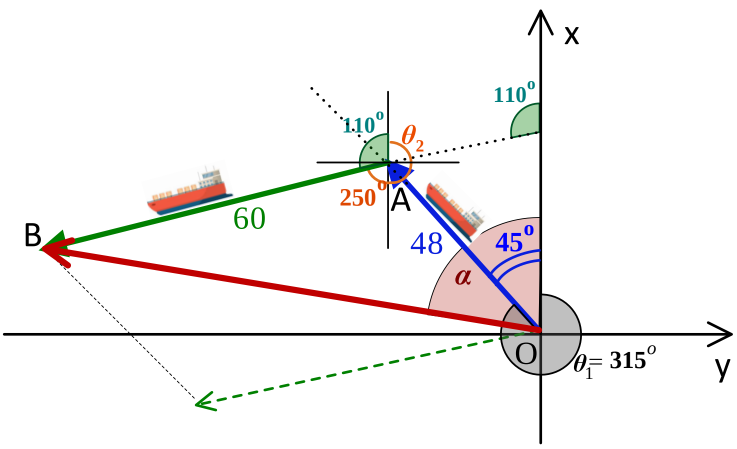



*Figure 52*. Illustration of Problem 5.

Solution

We use the standard navigation coordinate system in which the Ox-axis is directed toward north and the Oy-axis toward east.

Let the first displacement vectors corresponding to the first and second legs of the route be $\vec{s}_1$ and $\vec{s}_2$ . respectively.

The resultant displacement vector is the sum of $\vec{s}_1$ and $\vec{s}_2$ .

$$\vec{s} = \vec{s}_1 + \vec{s}_2$$

The distance from the port to the ship is equal to the magnitude of the resultant displacement vector$|\vec{s}|$.

Let us find the components of vectors $\vec{s}_1$ and $\vec{s}_2$.

1) Components of $\vec{s}_1$:

Since the ship leaves the port and sails 48 nautical miles on a bearing of 315°, the magnitude of the first displacement vector $|\vec{s}_1| = 48$ and the clockwise angle from north to the vector is $\theta_1 = 315°$ .

The components of the first displacement vector are (Fig.)

$$s_{1_x} = |\vec{s}_1|\, cos\theta_1 = 48\cos 315° = 48\, cos(360° - 315°) = 48\cos 45° \approx\; 33.94$$

$$s_{1_y} = |\vec{s}_1|\, sin\theta_1 = 48\sin 315° = 48\, sin\,(360° - 315°) = -48\sin 45° \approx\; -33.94$$

Thus,

$$\vec{s}_1 = (33.94, -33.94)$$

2) Components of $\vec{s}_2$:

The magnitude of the second displacement vector is $|\vec{s}_2| = 60$ and the clockwise angle from north to the vector is $\theta_2 = 250°$ , therefore the components of $\vec{s}_2$ are:

$$\vec{s}_{2_x} = |\vec{s}_2|\, cos\theta_2 = 60\cos 250° = 60\cos(360° - 250°) = 60\cos 110° \approx\; -20.52$$

$$\vec{s}_{2_y} = |\vec{s}_2|\, cos\theta_2 = 60\sin 250° = 60\sin(360° - 250°) = 60\sin 110° \approx\; -56.38$$

Hence,

$$\vec{s}_2 = (-20.52, -56.38)$$

3) The resultant displacement vector is

$$\vec{s} = \vec{s}_1 + \vec{s}_2 = (33.94, -33.94) + (-20.52, -56.38) = (13.42, -90.32)$$

3) The distance from the port to the ship is

$$|\vec{s}| = \sqrt{13.42^2 + (-90.32)^2} \approx 91.31 \text{ nautical miles}$$

4) The bearing $\theta$ would be determined as $= 360° - \alpha$, wehere $\alpha$ is the angle between vector $\vec{s}$ and OX-axes

$$\cos\alpha = \frac{s_x}{|\,\vec{s}\,|} = \frac{13.42}{91.31} \approx 0.1469 \Rightarrow \alpha = \arccos(0.1469) \approx 81.56°$$

The ship's bearing is

$$\theta = 360° - 81.56° = 278.44°$$

*Answer:* The resultant displacement vector of the ship is $\vec{s} = (13.42, -90.32)$; the distance from the port to the ship is $|\vec{s}| \approx 91.31$ nmi; the bearing of the ship is $\theta = 278.44°$

***Note:*** The distance from the port to the ship, OB, and the bearing of the ship's final position from the port can be determined from triangle OAB using the law of cosines and the law of sines, without resolving the vectors into their x- and y-components.

**Vectors in Orbital Mechanics:**

We consider a problem in orbital mechanics, illustrating the application of fundamental vector operations—vector magnitude, cross product, normalization, and dot product— the angle between two orbital planes, and the inclination of an orbit.

In orbital mechanics, calculations are commonly performed in the non-rotating Earth-Centered Inertial (ECI) coordinate system. The ECI frame is a Cartesian (*XYZ*) coordinate system defined as follows:

– the origin is located at the center of the Earth;

– the Z-axis coincides with the Earth's rotation axis and points toward the North Pole;

– the XY-plane coincides with the Earth's equatorial plane;

– the X-axis lies in the equatorial plane and points toward the vernal equinox;

– the Y-axis completes a right-handed coordinate system.

The geocentric position of a satellite is described by the radius vector

$$\vec{r} = (x, y, z),$$

which points from the Earth's center to the satellite (Fig.53).

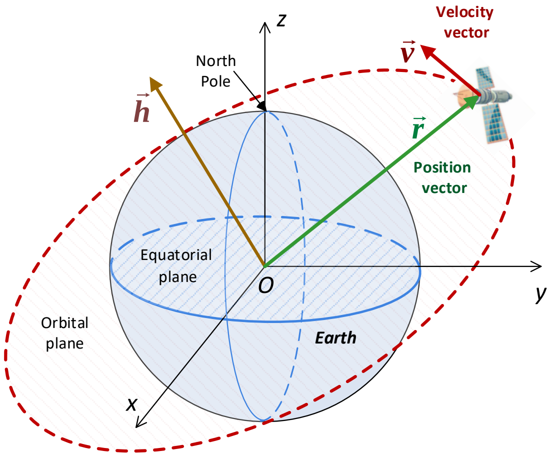



*Figure 53*. Orbit of a satellite around the Earth

The orbital plane is the geometric plane in which the satellite's trajectory lies. Its spatial orientation is determined by the specific angular momentum vector $\vec{h}$, which is perpendicular to the orbital plane at any instant and is defined by the vector product

$$\vec{h} = \vec{r} \times \vec{v},$$

where $\vec{v}$ is the geocentric inertial velocity vector of the satellite, directed tangentially to the orbit.

Problem 5.

Two artificial Earth satellites move along Keplerian orbits. At a given instant, their geocentric position and velocity vectors in the Earth-Centered Inertial (ECI) reference frame are:

Satellite 1

$$\vec{r}_1 = (6520, 1240, 1300)\ \mathrm{km}$$

$$\vec{v}_1 = (-1.464,\ -0.278,\ \ 7.605)\ \mathrm{km/s}$$

Satellite 2

$$\vec{r}_2 = (-2800, 5900, 2100)\ \mathrm{km}$$

$$\vec{v}_2 = (7.029, 2.960, 1.054)\ \ \mathrm{km/s}$$

Determine:

1) the normal vector to the orbital plane of each satellite,
2) the angle between the orbital planes (mutual inclination),
3) the inclination of each orbit.

Solution

1. Since the specific angular momentum vector $\vec{h}$ is perpendicular to the orbital plane, the corresponding unit normal vector is

$$\vec{n} = \frac{\vec{h}}{|h|}.$$

We find the specific angular momentum vector $\vec{h}$ and the unit normal vector to the orbital plane for each satellite.

Satellite 1:

$$\vec{h}_1 = \vec{r}_1 \times \vec{v}_1 = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 6520 & 1240 & 1300 \\ -1.464 & -0.278 & 7.605 \end{vmatrix} =$$

$$= \vec{\imath} \begin{vmatrix} 1240 & 1300 \\ -0.278 & 7.605 \end{vmatrix} - \vec{\jmath} \begin{vmatrix} 6520 & 1300 \\ -1.464 & 7.605 \end{vmatrix} + \vec{k} \begin{vmatrix} 6520 & 1240 \\ -1.464 & -0.278 \end{vmatrix}$$

After evaluation,

$$\vec{h}_1 = 9791.6\,\vec{\imath} - 51487.8\,\vec{\jmath} + 2.8\,\vec{k}$$

$$|\vec{h}_1| = \sqrt{9791.6^2 + (-51487.8)^2 + 2.8^2} \approx 52410.58$$

The corresponding unit normal vector to the orbital plane of the first satellite is

$$\vec{n}_1 = \frac{\vec{h}_1}{|\vec{h}_1|} = \frac{(9791.6\,, -51487.8,\ 2.8\,)}{52410.58} = (0.186825, -0.982393, 0.000053)$$

Satellite 2:

$$\vec{h}_2 = \vec{r}_2 \times \vec{v}_2 = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -2800 & 5900 & 2100 \\ 7.029 & 2.960 & 1.054 \end{vmatrix} =$$

$$= \vec{\imath} \begin{vmatrix} 5900 & 2100 \\ 2.960 & 1.054 \end{vmatrix} - \vec{\jmath} \begin{vmatrix} -2800 & 2100 \\ 7.029 & 1.054 \end{vmatrix} + \vec{k} \begin{vmatrix} -2800 & 5900 \\ 7.029 & 2.960 \end{vmatrix} =$$

$$= 2.6\,\vec{\imath} + 17712.1\,\vec{\jmath} - 49759.1\,\vec{k}$$

Thus,

$$\vec{h}_2 = 2.6\,\vec{\imath} + 17712.1\,\vec{\jmath} - 49759.1\,\vec{k}$$

$$|\vec{h}_2| = \sqrt{2.6^2 + 17712.1^2 + (-49759.1)^2} \approx 52817.48$$

The corresponding **unit normal vector** to the orbital plane of the second satellite is

$$\vec{n}_2 = \frac{\vec{h}_2}{|\vec{h}_2|} = \frac{(\,2.6\,, 17712.1\,, -49759.1\,)}{52817.48} = (0.000049, 0.335345, -0.942095)$$

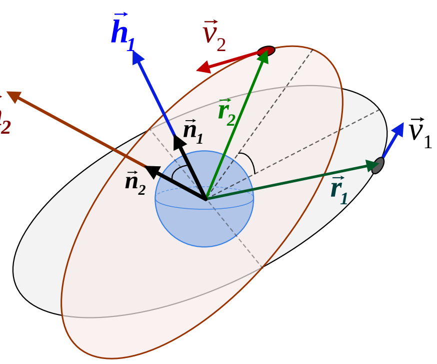



*Figure 54*. The angle between the orbital planes

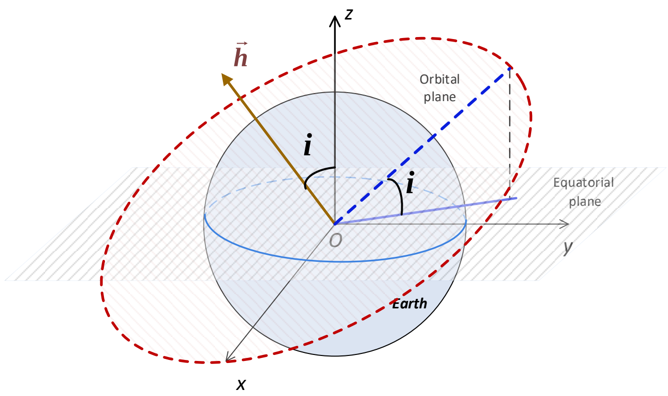



*Figure 55*. An inclination of the satellite orbit

2) The angle between the orbital planes equals the angle between their unit normal vectors $\vec{n}_1$ and $\vec{n}_2$ (Fig.54):

$$cos\Phi = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| \cdot |\vec{n}_2|} = \vec{n}_1 \cdot \vec{n}_2$$

$$cos\ \Phi = (0.186825, -0.982393, 0.000053) \cdot (0.000049, 0.335345, -0.942095) =$$

$$= 0.186825 \cdot 0.000049 + (-0.982393) \cdot 0.335345 + \ 0.000053 \cdot (-0.942095) = -0.32948$$

Hence, the angle between the orbital planes is

$$\Phi = \arccos(-0.32948) \approx 109.24^o$$

The minimum angle between the orbital planes is

$$\Phi_{min} = \arccos(|\vec{n}_1 \cdot \vec{n}_2|) = \arccos(0.32948) \approx 70.76^o$$

3) Inclination of each orbit

The inclination *i* is the angle between the orbital plane and the Earth's equatorial plane (Fig.55). It is equal to the angle between the angular momentum vector and the Z-axis.

$$cosi = \frac{h_z}{|\vec{h}|} = n_z$$

where $h_z$ is the z -component of the angular momentum vector $\vec{h}$, and $n_z$ is the z -component of the unit normal vector $\vec{n}$

The inclination of the orbital plane of Satellite 1 is:

$$i_1 = arccos\left(\frac{h_{1_z}}{|\vec{h}_1|}\right) = arccos(0.000053) \approx 89.997°$$

The inclination of the orbital plane of Satellite 2 is:

$$i_2 = arccos\left(\frac{h_{2_z}}{|\vec{h}_2|}\right) = arccos(-0.942095) \approx 160.5°$$

*Answer:* The minimum angle between the orbital planes is $\Phi \approx 70.76^o$.

The inclinations are $i_1 \approx 89.997°$ , $i_2 \approx 160.5°$.

**Vectors in computer graphics:**

Vectors constitute a fundamental mathematical tool in 3D computer graphics. They are used to represent the position, shape, and motion of objects, to model lighting and reflection phenomena within a virtual scene.
The following examples illustrate some of these applications.

The position of an object in three-dimensional space, relative to a chosen coordinate system (for example, the origin), is represented by a position vector $\vec{r}(t)$. Motion is described as a change in this vector over time.
For an object moving with constant velocity, the position at time t is given by:

$$\vec{r}(t) = \vec{r}_0 + \vec{V}\Delta t$$

where $\vec{\boldsymbol{r}}_{\boldsymbol{0}}$ is the initial position vector and $\vec{\boldsymbol{V}}$ is the constant velocity vector.

This relation forms the basis of object animation, trajectory simulation, and dynamic scene modelling.

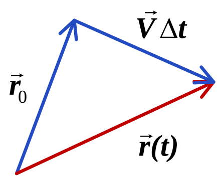



*Figure 56*. Position of an object at time t

Problem 6.

The initial position of the object is given by the radius vector $\vec{r}_0 = (2, 1, 0)$.
An object moves in three-dimensional space with constant velocity $\vec{V} = (3, -2, 1)$.
Determine the position of the object after the time interval $\Delta t = 2$ sec.

Solution

Let the new vector position is $\vec{r}_{new}$

$$\vec{r}_{new} = \vec{r}_0 + \vec{V}\,\Delta t$$

Substituting the given vectors

$$\vec{r}_{new} = (2, 1, 0) + 2(3, -2, 1) = (8, -3, 1)$$

*Answer*: After 2 seconds, the object will be at the point $(8, -3, 1)$.

*Note:* A similar formula is used in GPS navigation to calculate an object's current velocity, determine its direction of motion (for example, a car), and predict its future position.

To create realistic lighting in 3D graphics, including shadows, highlights, and reflections, the dot product of vectors is widely used, since it allows us to determine the angle between the direction of the light and the surface normal (Fig.58).

For example, to model light reflection from a matte or rough surface (such as concrete, paper, or plaster), the diffuse lighting model is used, in which light is reflected uniformly in all directions.

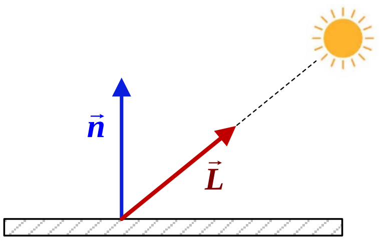



*Figure 58(a).*

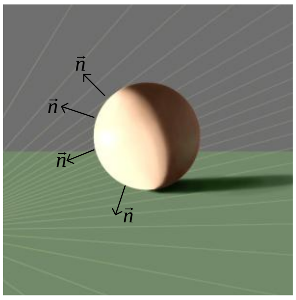



*Figure 58(b)*. Lighting in 3D graphics graphics

This model is described by Lambert's reflection model, according to which the brightness of an illuminated surface depends only on the angle of incidence of the light (Fig.58). The amount of light reaching the surface is given by the following formula:

$$I = k_d\, I_L\, \vec{L} \cdot \vec{n}$$

where

$I$ is the amount of light reaching the surface point (the diffuse lighting intensity at the surface point)
$\vec{L}$ is a unit vector towards the light source,
$\vec{n}$-is a unit normal vector to the surface (vector perpendicular to the surface at a given point)
$k_d$ is the diffuse reflection coefficient of the material (material property),
$I_L$ is the intensity of the light source

Only the surface orientation and the light direction are considered in this model.

If $\vec{L} \cdot \vec{n}$ =0, then the light strikes the surface at 90°, and the surface appears dark.

If $\vec{L} \cdot \vec{n}$ =1, then the surface is maximally illuminated.

Problem 7.

The surface normal vector at point P is $\vec{N} = (0, 1, 1)$.
The position of the surface point P is given by the radius vector $\vec{r}_P = (1, 2, 3)$ .
The position of the light source is given by the radius vector $\vec{r}_L = (4, 3, 6)$.
The diffuse reflection coefficient of the material is:$\mathrm{k_d} = 0.7$. The light intensity is $\mathrm{I_L} = 1$.
Compute the lighting intensity at point P.

Solution

Compute the light direction vector

$$\overrightarrow{PL} = \vec{r}_L - \vec{r}_P = (4, 3, 6) - (1, 2, 3) = (3,1, 3)$$

Compute its magnitude

$$\left|\overrightarrow{PL}\right| = \sqrt{3^2 + 1^2 + 3^2} = \sqrt{19}$$

Normalize the light vector

$$\vec{L} = \frac{\overrightarrow{PL}}{\left|\overrightarrow{PL}\right|} = \frac{(3,1, 3)}{\sqrt{19}} = \left(\frac{3}{\sqrt{19}}, \frac{1}{\sqrt{19}}, \frac{3}{\sqrt{19}}\right)$$

Normalize the surface normal vector:

$$\vec{n} = \frac{\vec{N}}{|\vec{N}|} = \frac{(0,1,1)}{\sqrt{2}} = \left(\frac{0}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Compute the dot product

$$\vec{L} \cdot \vec{n} = \left(\frac{3}{\sqrt{19}}, \frac{1}{\sqrt{19}}, \frac{3}{\sqrt{19}}\right) \cdot \left(\frac{0}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 0 + \frac{1}{\sqrt{38}} + \frac{3}{\sqrt{38}} = \frac{4}{\sqrt{38}}$$

Compute the diffuse lighting intensity I at point P:

$$I = k_d I_d \vec{L} \cdot \vec{n} = 0.7 \cdot 1 \cdot \frac{4}{\sqrt{38}} = \frac{2.8}{\sqrt{38}} \approx 0.454$$

*Answer*: $\mathrm{I} \approx 0.454$. It means that the surface receives approximately 45.4% of the maximum possible diffuse illumination.

To compute the pixel brightness in computer graphics, instead of the coefficients $k_d$ and $I_d$ in Lambert's cosine law, the surface color and the color of the incoming light are used (the RGB color model):

$$I = M\, I_L \max\left(0, \vec{L} \cdot \vec{n}\right)$$

where

$I$ is the brightness of the pixel
$\mathrm{M}$ is the surface color and reflectivity ( $0 \leq \mathrm{M} \leq 1$)
$\mathrm{I_L}$ is the color (brightness) of the incoming light

### 3.7. Exercises

1. Vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are shown in Figure 60.

Construct vectors:

a) $3\vec{c}$ b) $\frac{1}{3}\vec{c}$ c) $-2\vec{c}$

d) $\vec{a} + 2\vec{c}$ e) $\frac{1}{3}\vec{a} - \vec{c};$

g) $\vec{a} - \frac{1}{2}\vec{d}$ g) $2\vec{b} + \vec{c} - \frac{1}{3}\vec{a} + \vec{d}$

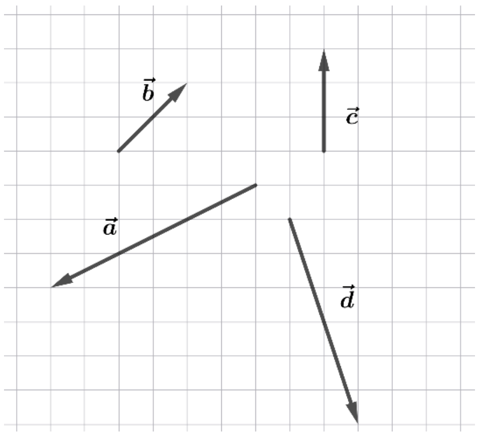


Figure 60.

2. In parallelogram ABCD, the vectors $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AD} = \vec{b}$ are given. Let M be the point of intersection of the diagonals. Express the vectors $\overrightarrow{MA}$, $\overrightarrow{MB}$, $\overrightarrow{MD}$, and $\overrightarrow{MC}$ in terms of vectors $\vec{a}$ and $\vec{b}$.

3. In triangle ABC, the vectors $\overrightarrow{\mathrm{AB}} = \vec{a}$ and $\overrightarrow{\mathrm{AC}} = \vec{b}$ are given. Let M, N, and P be the midpoints of sides BC, CA, and AB, respectively. Express the vectors $\overrightarrow{\mathrm{AM}}$, $\overrightarrow{\mathrm{BN}}$, and $\overrightarrow{\mathrm{CP}}$ in terms of $\vec{a}$ and $\vec{b}$.

4. The vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ form angles $2\pi/3, \pi/3$, and $\pi$, respectively, with a given axis $\vec{l}$. Find the scalar projection of the vector $\vec{a} + 3\vec{b} + 2\vec{c}$ onto the axis $\vec{l}$, if $|\vec{a}|$=2, $|\vec{b}|$=4, and $\vec{c}$ is a unit vector. (Use the properties of scalar projection)

5. Given two points M (2,-4,3) and N(-3,2,0), construct the vector $\overrightarrow{MN}$ in the coordinate system. Find its coordinates, magnitude, direction cosines, and a unit vector in the direction of the vector $\overrightarrow{MN}$.

6. Given vector $\overrightarrow{AB} = (6, -2, 1)$ and the initial point $A(-3;\ 1, 2)$. Find the coordinates of the endpoint *B.*

7. Given the vectors $\vec{a} = (-3, 1, -5)$ , $\vec{b} = (-1, -1, 5)$ and $\vec{c} = (6, 3, 9)$ . Find the coordinates, magnitude, and direction cosines of the vector $\vec{d} = 2\vec{a} - \vec{b} + \frac{1}{3}\vec{c}$ . Find the unit vector of the vector $\vec{d}$.

8. A vector $\vec{a}$ makes angles of $60^{o}$ and $45^{o}$ with the coordinate axes Ox and Oy, respectively. Its magnitude is $|\vec{a}| = 5$. Find its coordinates if the vector forms a wide angle with the Oz axis.

9. Given two vectors $\vec{a} = (4, -8, \alpha)$ and $\vec{b} = (\beta, 2, -3)$. Find all values of α and β for which the vectors $\vec{a}$ and $\vec{b}$ are collinear

10. Given the points A(−1,2,−3), B(1, −1, 0) and C(3, −4, 3). Prove that the points A, B, and C lie on the same straight line. (Use the collinearity of the vectors)

11. Given the points M (2, -3,−1), N(-2, −1, 4) and K($\alpha$, −9, $\beta$). Find values $\alpha$ and $\beta$ for which the point K lies on the straight line MN. (Use the collinearity of the vectors)

12. Given the points A(−4, 1, 0), B(4, 1, 2), C(2,−3, 4), and D(−2,−3, 2), prove that the quadrilateral ABCD is a trapezoid. Find the lengths of its sides. (Prove this by showing that one pair of opposite sides is parallel, but the other pair is not)

13. Given the vertices of quadrilateral ABCD: A(1, 0, 2),B(4, 1, 3),C(6, 4, 5),D(3, 3, 4). Prove that ABCD is a parallelogram.

14. Given three vertices of a parallelogram: A(−1, 2, 0), B(2, 3, 1), and C(5, 5, 3). Find the coordinates of the fourth vertex of the parallelogram.

15. Find a point on the Oz axis that is equidistant from points A(-1, 3,-2) and B(3,4,1).

16. Given $|\vec{a}| = 3\sqrt{2}$, $|\vec{b}| = 5$, and the angle between $\vec{a}$ and $\vec{b}$ be $\pi/4$. Find $\vec{a}\cdot\vec{b}$.

17. Given $|\vec{a}| = 4$, $|\vec{b}| = \sqrt{3}$, and the angle $\angle(\vec{a},\vec{b}) = 150^0$.

Find: a) $(2\vec{a}+\vec{b})\cdot(\vec{a}-4\vec{b})$ b) $(2\vec{a}+\vec{b})^2$

c) $|2\vec{a}+\vec{b}|$ d) $|\vec{a}-4\vec{b}|$

18. Given $|\vec{a}| = 1$, $|\vec{b}| = 4$, and the angle $\angle(\vec{a},\vec{b}) = \pi/3$.
Find the value $\beta$ for which the vectors $(\beta\vec{a}-3\vec{b})$ and $(5\vec{a}+2\vec{b})$ are perpendicular.

19. Given the vectors $\vec{a} = 2\vec{\imath}-4\vec{\jmath}+2\vec{k}$ and $\vec{b} = 3\vec{\imath}+5\vec{k}$
Find: a) $\vec{a}\cdot\vec{b}$;
b) $(\frac{1}{2}\vec{a}+\vec{b})\cdot(\vec{a}-4\vec{b})$

20. Determine whether the vectors $\overrightarrow{KL}$ and $\overrightarrow{MN}$ are perpendicular, given the points K, L, M, N:

a) K(-1,2,5), L(1;1;6) M(-1,0,7) N(0,1,6)

b) K(3,-1,1), L(2;4;5) M(1,6,-2) N(-1,0,-5)

21. Given the points $A(-1,\lambda,5)$, $B(2,4,1)$, $C(\lambda, 3, 0)$, and $C(4, 1, -1)$.

Find the value $\lambda$ for which the line segments AB and CD are perpendicular.

22. Given vectors $\vec{p} = (-1,3,1)$ and $\vec{q} = (2,-4, 3)$. Find the angle between the vectors $\vec{a} = 2\vec{p}-\vec{q}$ and $\vec{b} = \vec{p}+\vec{q}$.

23. Let us consider a parallelogram OACB whose adjacent sides coincide with the vectors $\overrightarrow{OA} = (1,1,0)$ and $\overrightarrow{OB} = (0,-3,2)$.

a) Express the diagonals of the parallelogram in vector form and determine their lengths;

b) Find the angle between the diagonals.

24. The vertices of the triangle ABC are given by

A(2, -3, 0), B(1, −1, 4), C(3, 1, 2)

Find: a) The interior angle at vertex B;
b) The median drawn from vertex B.

25. Given the points $A(-2,-2,5)$, $B(1,3,0)$,and $C(-1,-4,3)$. Find the scalar projection of the vector $\overrightarrow{AB}$ onto the vector $\overrightarrow{AC}$

26. Given the vectors $\vec{a} = (-3,5,-1)$, $\vec{b} = (-1,4,1)$, and the vector $\vec{c} = 4\vec{a}-\vec{b}$.
Find the scalar projection of $\vec{c}$ onto $\vec{b}$.

27. Given the vectors $\vec{a} = \vec{\imath}-3\vec{k}$, $\vec{b} = 3\vec{\imath}-2\vec{\jmath}+\vec{k}$, and $\vec{c} = -5\vec{\imath}+\vec{\jmath}-2\vec{k}$.
Find the scalar projection of $\vec{a}$ onto the vector $3\vec{b}+2\vec{c}$.

28. The following forces act upon a material point

$$\vec{F}_1 = 4\vec{\imath} + 2\vec{\jmath} + \vec{k}\,, \quad \vec{F}_2 = 2\vec{\imath} - \vec{\jmath} + 3\vec{k}.$$

Determine the work done by the resultant force when the material point moves in a straight line from the origin point of the coordinate system to the point A(3, 1, -2).

29. Let $|\vec{a}| = 3$, $|\vec{b}| = 2\sqrt{3}$, and the angle $\angle(\vec{a}, \vec{b}) = \pi/3$ be given. Find $|\vec{a} \times \vec{b}|$.

30. Simplify the following expressions:

a) $(\vec{p} - 4\vec{q}) \times (3\vec{p} - 2\vec{q})$

b) $(\vec{\imath} - \vec{\jmath}) \times (\vec{\imath} + 2\vec{\jmath}) + \vec{\jmath} \times (\vec{\imath} + \vec{\jmath} + \vec{k}) + (\vec{k} - \vec{\jmath}) \times (\vec{k} + \vec{\imath}).$

31. Given $\quad \vec{s} = (2\vec{a} + \vec{b}) \times (\vec{c} - \vec{a}) + (\vec{b} + \vec{c}) \times (\vec{a} + \vec{b})$,

$$|\vec{a}| = 5, \quad |\vec{c}| = 4\,, \ \angle(\vec{a}, \vec{c}) = 2\pi/3.$$

a) Simplify the expression for $\vec{s}$,

b) Find the magnitude of the vector $\vec{s}$.

32. Given $|\vec{a}| = \sqrt{2}$, $\ |\vec{b}| = 5$, $\angle(\vec{a}, \vec{b}) = \pi/4$. Find $\left|\, \vec{a} \times \vec{b} + (4\vec{a} + 3\vec{b}) \times (\vec{a} - 2\vec{b})\right|$.

33. Adjacent sides of a parallelogram are given by the vectors $\vec{a} = 2\vec{m} + \vec{n}$ and $\vec{b} = \vec{m} - 3\vec{n}$. Find the area of the parallelogram if $|\vec{m}| = 3$, $\ |\vec{n}| = 4$, and $\angle(\vec{m}, \vec{n}) = \frac{5\pi}{6}$.

34. The adjacent sides of a parallelogram are given by the vectors

$\overrightarrow{OA} = 2\vec{a} + \vec{b}$ and $\overrightarrow{OB} = \vec{a} - \vec{b}$, where $|\vec{a}| = 1$, $\ |\vec{b}| = 3$, and $\angle(\vec{a}, \vec{b}) = \frac{2\pi}{3}$.

Find: a) The area of the parallelogram

b) The lengths of its diagonals

35. Given the vectors $\vec{a} = 2\vec{\imath} + \vec{\jmath} - \vec{k}$, $\vec{b} = -3\vec{\imath} + 6\vec{k}$.

Find a) the vector $\vec{a} \times \vec{b}$ and its magnitude

b) the vector $(\vec{a} + \frac{1}{3}\vec{b}) \times (\vec{a} - \vec{b})$ and its magnitude.

36. Consider a parallelogram OADB whose adjacent sides coincide with the vectors $\overrightarrow{OA} = (-3,2,-1)$ and $\overrightarrow{OB} = (5,0,4)$. Find the area of the parallelogram.

37. Given three vertices of a parallelogram ABCD: A(-2,0,1), B(1,1,2) C(0,-1,3) .
Find: a) the area of the parallelogram (using the cross product)
b) the angle at the vertex B (using the dot product)

38. Given the coordinates of the vertices of the triangle ABC: A(-2, 4, 5), B(4, 1, 3), C(-1, 3, 2). Find its area.

39. Find the area of triangle ABC and the length of the altitude from the vertex A, if the coordinates of its vertices are A(3, 0, -1), B(2, 3, -1), C(0, 4, 1) .

40. Three forces $\vec{F}_1 = -\vec{\imath} + 3\vec{\jmath} + \vec{k}$ , $\vec{F}_2 = 2\vec{\imath} + 4\vec{\jmath} + 2\vec{k}$ , $\vec{F}_3 = \vec{\imath} - \vec{\jmath} + \vec{k}$ are applied at point M(-2,1,4). Find the moment of the resultant force about point K(1,-2,-3).

41. Check whether the given vectors are coplanar:

a) $\vec{a} = (1, 2, -1)$, $\vec{b} = (2, -1, 1)$, $\vec{c} = (3, 1, 0)$

b) $\vec{a} = (3, 2,1)$, $\vec{b} = (-3, 2, -1)$, $\vec{c} = (3, 1, 4)$

42. Check whether the given points lie in one plane.

a) $M_1(0, 2, -6)$, $M_2(-2, 0, 1)$, $M_3(1, -1, 1)$ , $M_4(-1, 2, 8)$

b) $\mathrm{A}(0, 2, -6)$, $B(-2, 0, 1)$, $C(1, -1, 1)$ , $D(-1, 2, 8)$

43. A parallelepiped is constructed on the vectors $\vec{a} = -\vec{\imath} + +2\vec{\jmath} - 3\vec{k}$, $\vec{b} = \vec{\imath} - \vec{k}$ and $\vec{c} = 4\vec{\imath} + 2\vec{\jmath} + 5\vec{k}$ . Find its volume.

44. Find the volume of the triangular pyramid ABCD if the coordinates of the vertices are $\mathrm{A}(3, -2,5)$, $B(0, -2, 5)$, $C(4, 0,0)$ , and $D(5, -3, 2)$

45. Points O(-5, -4, 3), A(4 ,3,1), B(2, 1, 2), and C(1,3,4) are the vertexes of the triangular pyramid OABC . Find the volume of the pyramid and the height drawn from the vertex O.

*Applications:*

46. A tugboat pushes a cruise ship with force $\vec{F} = 2.1\vec{\imath} + 1.3\vec{\jmath}$ MN, moving the ship along a straight path with displacement $\vec{r} = 400\vec{\imath} + 520\vec{\jmath}$

Find (a) The work done by the tugboat
(b)The angle between the force and displacement

47. Two tugboats tow a large ship in a harbour. The first tugboat exerts a force $F_1 = 2$ MN directed 20° East of North, and the second tugboat exerts a force $F_2 = 1.7$ MN directed 15° East of South. The ship moves 400 m northward (*A coordinate system is chosen such that the x-axis points east and the y-axis points north*).

Find (a) the resultant force acting on the ship.
a) The magnitude of the resultant force.
b) The work done by the resultant force if the ship during the displacement.

48. A rigid body in the shape of a rectangular parallelepiped is shown in Fig.61. A force of magnitude 300N is applied at the point A and is directed along line AB.
(a) Represent the force in Cartesian component form.

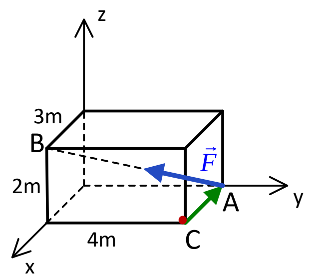


Figure 60. Exercise Nr. 48.

(b) Using the vector cross product, determine the moment of force $\vec{\mathrm{F}}$ about point C and its magnitude.

49. During a ground test of an aircraft control surface, a technician applies a horizontal force to the edge of an aileron to measure the torque acting on its hinge axis. The force vector and the position of the application point (relative to the hinge axis origin) are given as:

$$\vec{F} = (0{,}500{,}0)\ \text{N}, \vec{r} = (3{,}0{,}1)\ \text{m}.$$

(Here: the $x$-axis runs along the wingspan of the aircraft, the $y$-axis points toward the tail, the $z$-axis is directed upward). Find the moment of this force with respect to the hinge axis (the origin). Indicate both the magnitude and direction of the resulting moment vector.

50. A straight conductor segment of length 15 cm carries a current of 4 A. It is placed in a uniform magnetic field with a magnetic flux density vector $\vec{B} = -0.4\,\vec{\imath} + 0.3\vec{\jmath} + 0.1\,\vec{k}\ \mathrm{T}$. The conductor is oriented along the direction of the vector $\overrightarrow{MK}$ (current flows from M to K). Find the force acting on the conductor and its magnitude, if $\mathrm{M}(2{,}0, 1)$ and $\mathrm{K}(-1{,}4, 2)$

51. A particle with charge $\mathrm{q} = 2.1 \cdot 10^{-19}$ C enters a uniform magnetic field with velocity $\vec{v} = (2 \cdot 10^{6}, 0, 10^{6})$ m/s . The magnetic flux density is $\vec{\mathrm{B}} = 0.02\vec{\jmath} + 0.1\vec{k}\ \ \mathrm{T}$. Find the Lorentz force acting on the particle and its magnitude.

52. A cargo ship leaves a port and sails 50 nautical miles on a bearing of 060°. It then changes course and sails 48 miles on a bearing of 135°.

Determine a) the ship's resultant displacement vector,
b) the distance from the port to the ship
c) the bearing of the ship's final position from the port.

53. An aircraft takes off from an airport and flies 500 km on a bearing of 045°. It then changes course and flies 1200 km on a bearing of 105°. Finally, it changes course again and flies 1000 km nautical miles on a bearing of 210°.

Determine:
a) the aircraft's resultant displacement vector (in north–east components),
b) the distance from the airport to the aircraft,
c) the bearing of the aircraft's final position from the airport.

54. A cargo ship maintains a speed of 18.41 mph through the water on a heading of 055° (true). An ocean current set toward 120° at 2.30 mph. A strong crosswind produces an additional drift of 0.92 mph toward 090° (due east). Find the ship's ground-velocity vector (north–east components and magnitude) and its course over ground (bearing from north).

55. An artificial Earth satellite moves along Keplerian orbits. At a given moment, its geocentric position and velocity vectors in the Earth-Centered Inertial (ECI) reference frame are:

$$\vec{\boldsymbol{r}} = (\mathbf{5480}, \mathbf{3120}, \mathbf{4210})\text{km} \quad \text{and} \quad \vec{\boldsymbol{v}} = \frac{(\mathbf{-4.215{,}5.982{,}2.146})\,\mathbf{km}}{\mathbf{s}}$$

Determine:

(a) the specific angular momentum vector $\vec{h}$

(b) The normal unit vector to the orbital plane of the satellite,

(c) The inclination of the satellite orbit.

56. Two artificial Earth satellites move along Keplerian orbits. At a given instant, their geocentric position and velocity vectors in the Earth-Centered Inertial (ECI) reference frame are:

Satellite 1

$$\vec{\boldsymbol{r}}_{\mathbf{1}} = (\mathbf{6466}, \mathbf{3749}, -\mathbf{273})\text{km} \quad \text{and} \quad \vec{\boldsymbol{v}}_{\mathbf{1}} = (-\mathbf{4.160}, \mathbf{5.777}, -\mathbf{3.383})\text{ km/s}$$

Satellite 2

$$\vec{\boldsymbol{r}}_{\mathbf{1}} = (\mathbf{5603}, \mathbf{419},\ -\mathbf{5260})\text{km} \quad \text{and} \quad \vec{\boldsymbol{v}}_{\mathbf{1}} = (\mathbf{1.914}, -\mathbf{7.245}, \mathbf{0.992})\ \mathbf{km/s}$$

Determine:

(a) the normal unit vector to the orbital plane of each satellite,

(b) the angle between the orbital planes (mutual inclination),

(c) the inclination of each orbit.

57. The initial position of the object is given by the radius vector $\vec{r}_0 = (2, 4, 5)$.
An object moves in three-dimensional space with constant velocity $\vec{V} = (2, 3, -2)$.
Determine the position of the object after the time interval $\Delta t = 5$ sec.

**3.8. Answers:**

2.) $\overrightarrow{MA} = -\frac{1}{2}(\vec{a} + \vec{b}), \overrightarrow{MB} = \frac{1}{2}(\vec{a} - \vec{b}), \overrightarrow{MD} = \frac{1}{2}(\vec{b} - \vec{a}), \overrightarrow{MC} = \frac{1}{2}(\vec{a} + \vec{b})$

3.) $\overrightarrow{AM} = \frac{1}{2}(\vec{a} + \vec{b}),\ \overrightarrow{BN} = \frac{1}{2}\vec{b} - \vec{a},\ \overrightarrow{CP} = \frac{\vec{a}}{2} - \vec{b}$ 4.) *3* 5.) (-5,6, -3); $\sqrt{70}$,
$(-\frac{\sqrt{70}}{14}, \frac{3\sqrt{70}}{35}, -\frac{3\sqrt{70}}{70})$ 6.) $(3, -1, 3)$ 7.) $\vec{c} = (-3, 4, -12),\ |\vec{c}| = 13, (-\frac{3}{13}, \frac{4}{13}, -\frac{12}{13})$

8.) $\vec{a} = \left(\frac{5}{2}, \frac{5\sqrt{2}}{2}, -\frac{5}{2}\right)$ 9.) $\alpha = 12, \beta = -1$ 11.) $\alpha = 14,\ \beta = -16$

12) $AB = 2\sqrt{17}, BC = 2\sqrt{6},\ CD = 2\sqrt{5},\ DA = 2\sqrt{6}$ 14) D (2,4,2) 15.) (0,0,2) 16.) 15

17.) (a) 62; (b) 43; (c)$\sqrt{43}$; (d) $4\sqrt{7}$ 18.) *14* 19.) (a) 16; (b) -140

20.) (a) perpendicular, (b) not perpendicular 21.) 8 22.) $\arccos(-\sqrt{26}/13) \approx 113°$

23.) (a) *3;* $\sqrt{21}$ (b) $\arccos(11\sqrt{21}/63) \approx 36.86°$ 24.) (a) $\arccos\left(\frac{\sqrt{7}}{7}\right) \approx 67.79°$,

(b) $3\sqrt{5}/2$ 25.) 1 26.) $35\sqrt{2}/3$ 27.) $\frac{\sqrt{2}}{3}$ 28.) 11 29.) 9 30.) *(a)* $10\vec{p}\times\vec{q}$ *, (b)* $\vec{j}+3\vec{k}$

31.) (a) $\vec{s}=\vec{a}\times\vec{c}$ , (b) $|\vec{s}|=10\sqrt{3}$ 32.) 50 33.) 42 34.) (a) $\frac{9\sqrt{3}}{2}$, (b) 3 and $\sqrt{31}$

35). (a) $6\vec{\imath}-9\vec{\jmath}+3\vec{k}$; $3\sqrt{14}$ , (b) $-8\vec{\imath}+12\vec{\jmath}-4\vec{k}$ ; $4\sqrt{14}$ 36) $\sqrt{213}$

37.) $5\sqrt{2}$; $arccos\left(\frac{2\sqrt{66}}{33}\right)$ 38.) $\frac{\sqrt{314}}{2}$ 39.) $\frac{\sqrt{65}}{2}$; $\frac{\sqrt{65}}{3}$ 40.) $-30\vec{\imath}+26\vec{\jmath}-24\vec{k}$

41.) (a) coplanar, (b) are not coplanar 42) (a) do not lie in one plane, (b) lie in one plane

43.) 26 44.) 11/2 45.) 7/2 ; 7/3

Applications:

46.) (a) 1516; (b) 20.3° 47.) (a) $1.124\vec{\imath}+0.237\vec{\jmath}$ ; (b) 1.15; (c) $9.5\cdot 10^{7}$ J

48.) (a) $\left(\frac{900}{\sqrt{29}},-\frac{1200}{\sqrt{29}},\frac{600}{\sqrt{29}}\right)$; (b) $\left(0,-\frac{1800}{\sqrt{29}},\frac{3600}{\sqrt{29}}\right)$; 747.2 49.) (−500,0,1500); 1581 N·m

50.) $\left(\frac{0.03\sqrt{26}}{13},\ -\frac{0.03\sqrt{26}}{13},\ \frac{0.21\sqrt{26}}{13}\right)$; 0.0841

51.) $-4.2\cdot 10^{-15}\vec{\imath}-4.2\cdot 10^{-15}\vec{\jmath}+8.4\cdot 10^{-15}\,\vec{k}$ ; $4.3\cdot 10^{-14}$

52.) (a) (-8.94, 77.24) *(supposing that OX-axis is North);* (b) 77.76; (c) 96.60°

53.) (a) (-823.05, 1012.66) (supposing that *OX*-axis is North); (b) 1304.96; (c) 129.1°

54.) 20.31 mph; 062.4°

55.) (a) $(-18488.70,-29505.23,45932.16)$ ; (b) $(-0.3208,-0.5119,0.7969)$ ; (c) 37.16°

56. ) (a) $\vec{n}_1\approx(-0.1889,0.3914,0.9006)$, $\vec{n}_2\approx(-0.6480,-0.2711,-0.7118)$

(b) 51.34°; (c) $i_1\approx 25.76°$; $i_2\approx 135.38°$ 57.) $(12,19,-5)$