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Department of Physics and Mathematics

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A mathematical model for a class of networks in applications

SUMMARY OF THE DOCTORAL THESIS
for obtaining the Doctoral Degree of Mathematics (Ph.D.)
sub-branch “Differential equations”

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Comments are welcome. Send them to the secretary of the Promotion Council of mathematics, Parades street 1, Daugavpils, LV-5400, Tel. +371 26495316, e-mail: anita.sondore@du.lv

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1 General Information

Doctoral thesis contains 95 pages, 102 references, 123 figures, 6 tables.

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Doctoral thesis

Object of research: a system of ordinary differential equations of the second and higher orders, used in models of gene regulatory networks.

Aims of research: to obtain results on properties of a special system of ordinary differential equations, making emphasis on attracting sets, their locations, dependence on built-in parameters and types of interrelation between elements. Special attention is paid to evolution of the system and prediction of its future behaviours.

Research tasks:

- overview of low-dimensional systems of ordinary differential equations (ODE), used in models of genetic regulatory networks (GRN);
- collecting information on equilibria (critical points) of attracting nature in low-dimensional systems;
- studying the nature of attractive equilibria in two-dimensional (2D) and three-dimensional (3D) systems;
- derivation of formulas for calculating the characteristic numbers of critical points in 2D and 3D systems;
- finding attractors, other than equilibria, in three-dimensional (3D) systems;
- considering examples of 3D systems, which have attractors in the form of stable periodic trajectories;
- considering examples of 3D systems, which exhibit chaotic behaviour of solutions;
- work with programs, detecting chaotic behavior on the basis of analysis of the Lyapunov exponents;
- considering systems of order four (4D), formulas for characteristic numbers of critical points;
- constructing examples of 4D systems, which have attractors in the form of stable equilibria;
- constructing examples of 4D systems, which have attractors in the form of stable periodic trajectories;

- considering examples of 4D systems, which exhibit chaotic behaviour of solutions;
- visualization of 4D attractors by projecting them on low-dimensional subspaces of the 4D phase space;
- considering examples of neuronal networks and detecting similarity in the corresponding ODE-type models;
- construction of examples of 5D and 6D systems which possess periodic attractors;
- considering examples of 6D systems, which exhibit irregular behaviour of solutions;
- visualization of attractors of 5D and 6D systems by projecting them into a lower dimension subspaces and considering graphs of components of solutions;
- overview of the results and outlining directions of future research.

Methods of research:

- classical techniques of mathematical analysis;
- comparison method;
- phase plane and phase space method;
- method of linearization around the trivial solution;
- perturbation method.

Main results: the results of the work were published in 23 scientific papers ([4], [48], [57]-[77]). Six of them ([4], [59], [67], [68], [69], [70]) have been published in the journals indexed in SCOPUS, three of them ([57], [66], [65]) were submitted to publish in the journals indexed in SCOPUS and two ([58], [71]) were submitted to publish in Web of Science journals. The results were communicated at several conferences of different levels:

1. Inna Samuilik, Nullcline method for research of GRN system critical points, The 78th Scientific Conference of the University of Latvia, (Riga, Latvia, February 28, 2020)
2. Felix Sadyrbaev, Svetlana Atslega, Inna Samuilik, On Controllability in Models of Biological Networks, VIII International Conference on Science and Technology, (Belgorod, Russia, September 24-25, 2020).
3. Inna Samuilik, Remark on four dimensional system arising in applications, The 79th Scientific Conference of the University of Latvia, (Riga, Latvia, February 26, 2021).
4. Felix Sadyrbaev, Inna Samuilik, Mathematical modelling of genetic regulatory networks, 2. International Baku Scientific Research Conference, (Baku, Azerbaijan, April 28-30, 2021).

5. Inna Samuilik, Felix Sadyrbaev, Mathematical modelling of evolution of multidimensional networks, 2. International Congress on Mathematics and Geometry, (Ankara, Turkey, May 20, 2021).

6. Svetlana Atslega, Felix Sadyrbaev, Inna Samuilik, On modelling of complex networks, 20th International Scientific Conference Engineering for Rural Development, (Jelgava, Latvia, May 27, 2021).

7. Inna Samuilik, Diana Ogorelova, Mathematical modelling of GRN using different sigmoidal functions, 1st International Symposium on Recent Advances in Fundamental and Applied Sciences, (Erzurum, Turkey, September 10-12, 2021).

8. Felix Sadyrbaev, Inna Samuilik, On the hierarchy of attractors in dynamical models of complex networks, 19th International Conference of Numerical Analysis and Applied Mathematics, (Rhodes, Greece, September 20-26, 2021)

9. Inna Samuilik, Felix Sadyrbaev, Valentin Sengileyev, Examples of periodic biological oscillators, International Conference "Differential Equations, Mathematical Modeling and Computational Algorithm", (Belgorod, Russia, October 25-29, 2021).

10. Felix Sadyrbaev, Inna Samuilik, Albert Silvans, On mathematical models of evolving networks, International Conference "Differential Equations and Related Topics, 24th joint session of Moscow Mathematical Society and I.G.Petrovskii Seminar", (Moscow, Russia, December 26-30, 2021).

11. Inna Samuilik, Felix Sadyrbaev, Diana Ogorelova, Mathematical modeling of three-dimensional genetic regulatory networks using different sigmoidal functions, International liberty interdisciplinary studies conference, (NewYork, ASV, January 16-17, 2022).

12. Inna Samuilik, On a four-dimensional system of differential equations related to the theory of gene regulatory networks, The 80th Scientific Conference of the University of Latvia, (Riga, Latvia, February 25, 2022).

13. Inna Samuilik, Felix Sadyrbaev, A Note on Attractor Selection, The 5th International Conference on Networking, Intelligent Systems and Security, (Bandung, Indonesia, March 30-31, 2022).

14. Inna Samuilik, Felix Sadyrbaev, Svetlana Atslega, Mathematical modelling of nonlinear dynamic systems, 21st International Scientific Conference Engineering for Rural Development, (Jelgava, Latvia, May 25-27, 2022).

15. Inna Samuilik, Genetic engineering-construction of a network of four dimensions with a chaotic attractor, 58th International JVE Conference,(Ventspils, Latvia, August 25-26, 2022).

2 Preface

The Theory of ordinary differential equations (ODE in short) has emerged from applications and serves applications. Closer to our times, new branches of the theory have appeared. Among them, the theory of boundary value problems (BVP) for ODE took significant place. Riga and its universities have strong traditions in this field. Pierce Bohl is known for the creation of fixed point theorems for integral and differential equations. In the middle of the 20th century, Anatoliy Myshkis had arrived in Riga to teach students, among them were Yurii Klokov and Arnol'd Lepin. Y. Klokov and A. Lepin headed the scientific division which studied BVP and related problems. This research is still continued now in the Institute of Mathematics and Computer Science of the University of Latvia. Y. Klokov and A. Lepin had many doctorate students and descendants. Some of them are still actively working in the field of differential equations, applications, and mathematical modeling. Another direction in the theory of differential equations was established by Professor Linard Reizins. This direction was aimed at the qualitative studies of ordinary differential equations. The problems of structural stability, classification of critical points for higher order equtions were in the center of his and his student's studies. It appears that Riga and Latvia had and still have long-standing traditions in the field of the theory of ODE.

In 2015, when joining the group of Professor Alexander Shostak for the studies in the field of telecommunication networks in the framework of a Europen project, the group of mathematicians from Riga and Daugavpils was attracted by a new kind of problems, where ordinary differential equations were involved. In the theory of telecommunication networks very active group of Japanese mathematicians, among them, Yuki Koizumi, Masayuki Murata, and others, proposed to use in the design of telecommunication networks the principles of self-organization, that could be found in Nature. It was pointed out, that in living organisms in any cell there exists a genetic regulatory network (GRN in short), responsible, among others, for reactions to changes in the environment. It was emphasized, that the mathematical model for GRN, which uses a system of ODE, can be used also for the management and control of telecommunication networks. The so-called Virtual Network Topology was proposed for the organization of a set of lightpaths, in order to establish a mechanism for quick response and rearrangement of a telecommunication network in bad conditions. A system of ODE, governing this process of rearrangement, had attracted the attention of researchers in Institute of Mathematics and Computer Science, University of Latvia and Daugavpils University. It appears, that accumulated previous knowledge in the theory of ODE and experience in the studies of ODE, can be applied to the new kind of problems. It was the starting point of research in this direction.

The system in the center of these studies is not easy, but it is in some sense symmetrical. It contains n ordinary differential equations of the form $X' = F(WX - \theta) - VX$, where the vector X is for unknowns, F is a vector of the so-called sigmoidal functions, W is $n \times n$ matrix (it is called regulatory one), θ and V are the parameters. This system will be denoted by S in this preface. It appeared first in the paper by Cowan-Vilson in the study of neuronal networks of the human brain. It was used to model genetic networks, and the meaning of X was different. It was associated with the protein expression of any gene. By protein expression genes communicate with each other. Affecting a single gene can affect the whole network. In system S the linear part describes the natural decay of a network, where there is no interrelation between genes. Some authors introduce in this model also other factors, such as stress. Generally speaking, the object of investigation is a multi-parameter autonomous quasi-linear system of ordinary differential equations. To the best of the author's knowledge, this system was not studied sufficiently for dimensions three and higher. One of the possible reasons is the lacking of theoretical results for systems of this kind. In the last decade, a number of papers had appeared, interpreting system S specifically. In the remarkable papers [10], [89] by Cornelius et al and Le-Zhi Wang et al. the X vector was interpreted as the state vector for the genetic network. This vector is time-dependent, $X(t)$, and it goes to its limiting set, or point. It is to be mentioned, that the phase space for the system S has a time-invariant set Q with the property: any trajectory of system S which enters Q, never escapes it. The existence of attracting sets in Q follows. In the above interpretation, some diseases can be treated having in mind that the respective state-vector $X(t)$ therefore is forced to go to the "wrong" attractor. Since system S contains a lot of parameters, some of them are adjustable and can be used to manage and control a network. Treatment of a disease then means redirecting of "bad" trajectory to a "normal" attractor.

The above considerations were good motivations for the study of system S and its attractor. The proposed promotional work contains achievements in this direction. Let us list them.

1. The 2-dimensional systems were studied, using the nullclines method;

2. The 3-dimensional systems were studied, using the nullcline method and extensive computational research; the main results are a) formulas for critical points of a 3D system; b) multiple examples of periodic attractors;

3. The 4-dimensional systems were studied, using previously obtained results for 2-dimensional systems; uncoupled 4D systems were constructed of two independent 2D-systems and various resulting combinations of attractors were studied; the main results are a) formulas for critical points; b) periodic attractors for uncoupled 4D systems; c) examples of periodic attractors; d) examples of perturbed 4D systems, which are no longer uncoupled; some conclusions were made about attractors in perturbed systems; d) an irregular behavior of solutions, tending to a 4D attractor, was observed;

4. Some examples of the 5-dimensional systems were examined;

5. The 6-dimensional systems were studied; the main results are a) examples of 6D-systems which were constructed of previously investigated three 2D systems; the resulting system can have attractors of periodic nature; b) examples of 6D-systems which were constructed of previously investigated two 3D systems; the resulting system can have attractors of periodic nature; c) examples of perturbed, and therefore coupled, 6D systems were examined; some observations on the behavior of solutions were made;

6. The 60-dimensional system was considered from the works [10],[89].

Generally, the work contains mainly computationally obtained results concerning systems of the form S, their phase space, examples of attractors and many related facts. The collection obtained results lay the foundation for further research into gene network models to understand them and develop methods of management and control.

3 Gene regulatory network

Gene regulatory networks (GRN in short) exist in any cell of any living organism. GRN regulates reactions to changes in the environment, controls the development of a cell, and manages the functioning of any kind. Elements of GRN, called genes, can influence other genes by sending proteins [43]. As a result of such influence, other genes can be activated or inhibited.
Attempts to mathematically model the functioning of GRN are multiple, using various mathematical objects and tools [26],[87]. To describe the evolution of a network, the most appropriate approach is using differential equations.

The typical system is of the form

$$X' = F(WX - \theta) - vX,$$

where X is the network state vector, F is a sigmoid nonlinearity with argument, transformed by multiplication with the regulatory matrix W, vX is a natural decay in absence of F.

Definition 3.1. *A dynamical system is a system of equations describing the time evolution of one or more dependent variables. Equations of motion can be modelled as differential equations and difference equations [34].*

Consider the n-dimensional dynamical system

$$\begin{cases} \dfrac{dx_1}{dt} = \dfrac{1}{1 + e^{-\mu_1(w_{11}x_1 + w_{12}x_2 + \ldots + w_{1n}x_n - \theta_1)}} - v_1 x_1, \\ \dfrac{dx_2}{dt} = \dfrac{1}{1 + e^{-\mu_2(w_{21}x_1 + w_{22}x_2 + \ldots + w_{2n}x_n - \theta_2)}} - v_2 x_2, \\ \ldots \\ \dfrac{dx_n}{dt} = \dfrac{1}{1 + e^{-\mu_n(w_{n1}x_1 + w_{n2}x_2 + \ldots + w_{nn}x_n - \theta_n)}} - v_n x_n, \end{cases} \tag{1}$$

where $\mu_n > 0$, θ_n and $v_n > 0$ are parameters and the coefficients w_{ij} are entries of the so called regulatory matrix

$$W = \begin{pmatrix} w_{11} & w_{12} & \ldots & w_{1n} \\ w_{21} & w_{22} & \ldots & w_{2n} \\ \ldots & & & \\ w_{n1} & w_{n2} & \ldots & w_{nn} \end{pmatrix}. \tag{2}$$

The parameters of the GRN have the following biological interpretations:

- v_i - degradation of the i-th gene expression product;
- w_{ij} - the connection weight or strength of control of gene j on gene i. Positive values of w_{ij} indicate activating influences while negative values define repressing influences;
- θ_i - influence of external input on gene i, which modulates the gene's sensitivity of response to activating or repressing influences.

4 Two-dimensional (2D) systems

Consider the system

$$
\begin{cases}
\dfrac{dx_1}{dt} = \dfrac{1}{1 + e^{-\mu_1(w_{11}x_1 + w_{12}x_2 - \theta_1)}} - v_1 x_1, \\[2ex]
\dfrac{dx_2}{dt} = \dfrac{1}{1 + e^{-\mu_2(w_{21}x_1 + w_{22}x_2 - \theta_2)}} - v_2 x_2,
\end{cases}
\tag{3}
$$

where μ_i and v_i are positive.

System (3) contains ten parameters $w_{ij}, \mu_i, \theta_i, v_i$. Changing any of these parameters can essentially affect the properties of the system and solutions. The construction of the characteristic equation is a nontrivial task.

The argument z of a sigmoidal function is transformed by the regulatory (coefficient) matrix

$$
W = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}. \tag{4}
$$

This matrix describes interrelation of elements x_i of a network. If GRN are studied, then the structure of W affects properties of the system and their solutions.

The nullclines are given by the equations

$$
\begin{cases}
x_1 = \dfrac{1}{v_1} \dfrac{1}{1 + e^{-\mu_1\,(w_{11}x_1 + w_{12}x_2 - \theta_1)}}, \\[2ex]
x_2 = \dfrac{1}{v_2} \dfrac{1}{1 + e^{-\mu_2\,(w_{21}x_1 - w_{22}x_2 - \theta_2)}}.
\end{cases}
\tag{5}
$$

The function $f(z) = \dfrac{1}{1 + e^{-\mu z}}$ is a sigmoid and it has the range of values $(0, 1)$. Therefore the first nullcline is in the strip $\{(x_1, x_2) : 0 < x_1 < \frac{1}{v_1}, x_2 \in \mathbb{R}\}$ and the second one is in the strip $\{(x_1, x_2) : x_1 \in \mathbb{R}, 0 < x_2 < \frac{1}{v_2}\}$. Therefore all critical points are located in the rectangle $Q := \{(x_1, x_2) : 0 < x_1 < \frac{1}{v_1}, 0 < x_2 < \frac{1}{v_2}\}$.

Proposition 4.1. *There exists at least one critical point for the system* (3)*.*

4.1 Linearized system

For the analysis of critical points, we need the linearized system. It takes the form

$$
\begin{cases}
u_1' = -v_1 u_1 + \mu_1 w_{11} g_1 u_1 + \mu_1 w_{12} g_1 u_2, \\
u_2' = -v_2 u_2 + \mu_2 w_{21} g_2 u_1 + \mu_2 w_{22} g_2 u_2,
\end{cases}
$$

where

$$g_1 = \frac{e^{-\mu_1(w_{11}x_1^* + w_{12}x_2^* - \theta_1)}}{[1 + e^{-\mu_1(w_{11}x_1^* + w_{12}x_2^* - \theta_1)}]^2},$$

$$g_2 = \frac{e^{-\mu_2(w_{21}x_1^* + w_{22}x_2^* - \theta_2)}}{[1 + e^{-\mu_2(w_{21}x_1^* + w_{22}x_2^* - \theta_2)}]^2}$$

and (x_1^*, x_2^*) is a critical point under consideration.

$$A = \begin{vmatrix} \mu_1 w_{11} g_1 - v_1 & \mu_1 w_{12} g_1 \\ \mu_2 w_{21} g_2 & \mu_2 w_{22} g_2 - v_2 \end{vmatrix}$$

$$A - \lambda I = \begin{vmatrix} \mu_1 w_{11} g_1 - v_1 - \lambda & \mu_1 w_{12} g_1 \\ \mu_2 w_{21} g_2 & \mu_2 w_{22} g_2 - v_2 - \lambda \end{vmatrix}$$

and the characteristic equation is

$$\begin{aligned}
&\det|A - \lambda I| = (\mu_1 w_{11} g_1 - v_1 - \lambda)(\mu_2 w_{22} g_2 - v_2 - \lambda) - (\mu_2 w_{21} g_2)(\mu_1 w_{12} g_1) = \\
&\mu_1\mu_2 w_{11} w_{22} g_1 g_2 - \mu_1 w_{11} g_1 v_2 - \mu_1 w_{11} g_1 \lambda - \mu_2 w_{22} g_2 v_1 + v_1 v_2 + v_1\lambda - \mu_2 w_{22} g_2 \lambda + \\
&v_2\lambda + \lambda^2 - \mu_1\mu_2 w_{12} w_{21} g_1 g_2 = \lambda^2 + (v_1 + v_2 - \mu_1 w_{11} g_1 - \mu_2 w_{22} g_2)\lambda + \\
&\mu_1\mu_2 w_{11} w_{22} g_1 g_2 - \mu_1 w_{11} g_1 v_2 - \mu_2 w_{22} g_2 v_1 - \mu_1\mu_2 w_{12} w_{21} g_1 g_2 + v_1 v_2 = 0.
\end{aligned}$$

To simplify we can write the characteristic equation as

$$\lambda^2 + B\lambda + C = 0, \tag{6}$$

where

$$B = v_1 + v_2 - \mu_1 w_{11} g_1 - \mu_2 w_{22} g_2,$$

$$C = \mu_1\mu_2 w_{11} w_{22} g_1 g_2 - \mu_1 w_{11} g_1 v_2 - \mu_2 w_{22} g_2 v_1 - \mu_1\mu_2 w_{12} w_{21} g_1 g_2 + v_1 v_2.$$

4.2 Critical points

For a simple continuous-time model, depending on the parameters, the attractor can be a single point (a critical point), two points (a two-point cycle), four, eight, or a larger finite number of points (a more complex cycle), a closed curve, or a chaotic attractor [23].

Definition 4.1. *An attractor is the limiting trajectory of the representing point in the phase space, to which all initial modes tend [3].*

Definition 4.2. *A limit cycle is a closed trajectory in phase space having the property that at least one other trajectory spirals into it, either as time approaches to infinity or as time approaches to negative infinity [17].*

Proposition 4.2. *A limit cycle can exist in nonlinear systems of ODE, the number of equations in which is $n \geq 2$.*

Proposition 4.3. *A closed trajectory has a critical point in its interior in space $\mathbb{R}^2$.*

If it is a stable state of equilibrium (critical point), the attractor of the system will be just a fixed point. If it is a stable periodic motion, then the attractor will be a closed curve, called the limit cycle [3].

4.3 Examples

Consider (6) and

$$B^2 < 4C, \; -\frac{B}{2} > 0 \Rightarrow B < 0.$$

Proposition 4.4. *If $w_{11} = w_{22} = 0$, then $B > 0$ and no critical point is an unstable focus.*

$$\begin{gathered}
B^2 - 4C = (v_1 + v_2 - \mu_1 w_{11} g_1 - \mu_2 w_{22} g_2)^2 - \\
-4(\mu_1\mu_2 w_{11} w_{22} g_1 g_2 - \mu_1 w_{11} g_1 v_2 - \mu_2 w_{22} g_2 v_1 - \mu_1\mu_2 w_{12} w_{21} g_1 g_2 + v_1 v_2) = \\
= v_1^2 + 2v_1 v_2 - 2v_1\mu_1 w_{11} g_1 - 2v_1\mu_2 w_{22} g_2 + v_2^2 - 2v_2\mu_1 w_{11} g_1 - 2v_2\mu_2 w_{22} g_2 \\
+\mu_1^2 w_{11}^2 g_1^2 + 2\mu_1 w_{11} g_1 \mu_2 w_{22} g_2 + \mu_2^2 w_{22}^2 g_2^2 - 4\mu_1\mu_2 w_{11} w_{22} g_1 g_2 + 4\mu_1 w_{11} g_1 v_2 \\
+4\mu_2 w_{22} g_2 v_1 + 4\mu_1\mu_2 w_{12} w_{21} g_1 g_2 - 4v_1 v_2 = \\
= (-v_1 + v_2 + \mu_1 w_{11} g_1)^2 + 2w_{22} g_2(\mu_2(v_1 - v_2 - \mu_1 w_{11} g_1) + 2\mu_1 w_{12} w_{21} g_1) + \mu_2^2 w_{22}^2 g_2^2.
\end{gathered}$$

Example 1. Consider $\mu_1 = \mu_2 = 10$, $v_1 = v_2 = 1$ and $\theta_1 = 1.2$, $\theta_2 = -0.7$. The regulatory matrix is

$$W = \begin{pmatrix} 0.5 & 2 \\ -2 & 0.5 \end{pmatrix}. \tag{7}$$

The characteristic equation for the critical point $(0.47; 0.47)$ is (6), where $B = -0.48$, $C = 24.66$.

Solving the equation we have $\lambda_1 = 0.2474 - 4.96i$ and $\lambda_2 = 0.2474 + 4.96i$. The type of the critical point is an unstable focus. The periodic solution emerges.

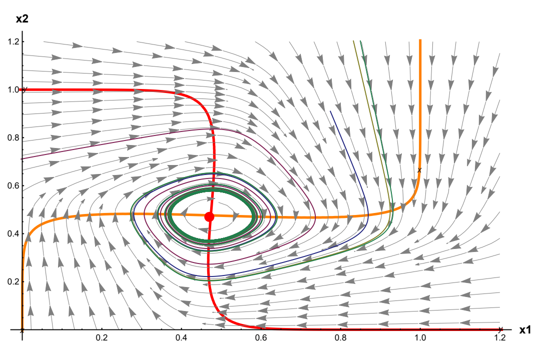

Figure 1: The phase portrait for the system (3) with the regulatory matrix (7). The type of the critical point is an unstable focus.

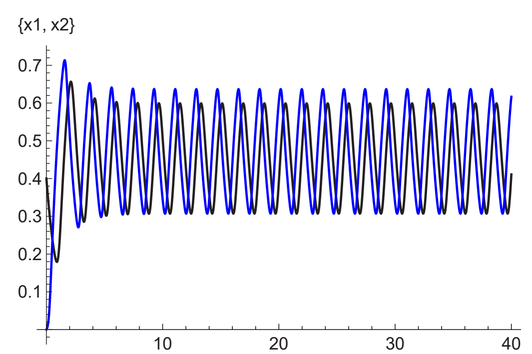

Figure 2: Solutions $(x_1(t), x_2(t))$ for the system (3) with the regulatory matrix (7).

Proposition 4.5. *Two-dimensional system of differential equations* (3) *can have nine critical points if* $w_{11}^2 + w_{22}^2 > 0$.

Example 2. Consider $\mu = 40$, $v_1 = v_2 = 1$ and $\theta_1 = \theta_2 = 2.5$. The regulatory matrix is

$$W = \begin{pmatrix} 5 & 2.2 \\ 2 & 3 \end{pmatrix}. \tag{8}$$

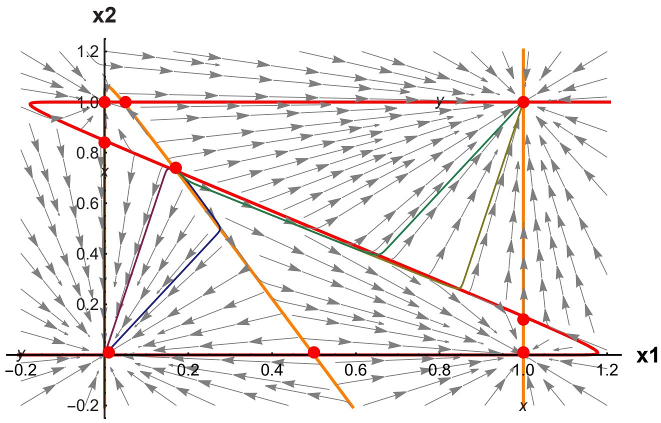

Figure 3: The phase portrait for the system (3) with the regulatory matrix (8). Nine critical points.

Proposition 4.6. *Suppose that elements w_{11} and w_{22} of the regulatory matrix (4) are zeros. Then the maximal number of equilibria in system (3) is three. Exactly one and exactly two critical points are possible.*

Proposition 4.7. *Suppose that elements w_{11} and w_{22} of the regulatory matrix (4) are not zeros and elements w_{12} and w_{21} are of opposite signs. Then the Hopf bifurcation may occur and the system (3) may have a limit cycle.*

Proposition 4.8. *Periodic solutions in system (4) cannot exist if* $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \neq 0$, *where f_1 and f_2 are the right sides of the equations in (4).*

5 Three-dimensional (3D) systems

Let us consider the system

$$\begin{cases} \dfrac{dx_1}{dt} = \dfrac{1}{1 + e^{-\mu_1(w_{11}x_1 + w_{12}x_2 + w_{13}x_n - \theta_1)}} - v_1 x_1, \\[2ex] \dfrac{dx_2}{dt} = \dfrac{1}{1 + e^{-\mu_2(w_{21}x_1 + w_{22}x_2 + w_{23}x_n - \theta_2)}} - v_2 x_2, \\[2ex] \dfrac{dx_3}{dt} = \dfrac{1}{1 + e^{-\mu_3(w_{31}x_1 + w_{32}x_2 + w_{33}x_3 - \theta_3)}} - v_3 x_3, \end{cases} \tag{9}$$

where μ_i, θ_i and v_i are the parameters, w_{ij} are the coefficients of the so-called regulatory matrix

$$W = \begin{pmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{pmatrix}. \tag{10}$$

The nullclines and the critical points for the system are defined by the relations

$$\begin{cases} x_1 = \dfrac{1}{v_1} \dfrac{1}{1 + e^{-\mu_1\,(w_{11}x_1 + w_{12}x_2 + w_{13}x_3 - \theta_1)}}, \\[2ex] x_2 = \dfrac{1}{v_2} \dfrac{1}{1 + e^{-\mu_2\,(w_{21}x_1 - w_{22}x_2 + w_{23}x_3 - \theta_2)}}, \\[2ex] x_3 = \dfrac{1}{v_3} \dfrac{1}{1 + e^{-\mu_2\,(w_{21}x_1 + w_{22}x_2 + w_{33}x_3 - \theta_3)}}. \end{cases}$$

5.1 Linearized system

The linearized system for any critical point (x_1^*, x_2^*, x_3^*) is

$$\begin{cases} u_1' = -v_1u_1 + \mu_1w_{11}g_1u_1 + \mu_1w_{12}g_1u_2 + \mu_1w_{13}g_1u_3, \\ u_2' = -v_2u_2 + \mu_2w_{21}g_2u_1 + \mu_2w_{22}g_2u_2 + \mu_2w_{23}g_2u_3, \\ u_3' = -v_3u_3 + \mu_3w_{31}g_3u_1 + \mu_3w_{32}g_3u_2 + \mu_3w_{33}g_3u_3, \end{cases}$$

where

$$g_1 = \frac{e^{-\mu_1(w_{11}x_1^* + w_{12}x_2^* + w_{13}x_3^* - \theta_1)}}{[1 + e^{-\mu_1(w_{11}x_1^* + w_{12}x_2^* + w_{13}x_3^* - \theta_1)}]^2}, \tag{11}$$

$$g_2 = \frac{e^{-\mu_2(w_{21}x_1^* + w_{22}x_2^* + w_{23}x_3^* - \theta_2)}}{[1 + e^{-\mu_2(w_{21}x_1^* + w_{22}x_2^* + w_{23}x_3^* - \theta_2)}]^2}, \tag{12}$$

$$g_3 = \frac{e^{-\mu_3(w_{31}x_1^* + w_{32}x_2^* + w_{33}x_3^* - \theta_3)}}{[1 + e^{-\mu_3(w_{31}x_1^* + w_{32}x_2^* + w_{33}x_3^* - \theta_3)}]^2}. \tag{13}$$

One has

$$A - \lambda I = \begin{vmatrix} \mu_1w_{11}g_1 - v_1 - \lambda & \mu_1w_{12}g_1 & \mu_1w_{13}g_1 \\ \mu_2w_{21}g_2 & \mu_2w_{22}g_2 - v_2 - \lambda & \mu_2w_{23}g_2 \\ \mu_3w_{31}g_3 & \mu_3w_{32}g_3 & \mu_3w_{33}g_3 - v_3 - \lambda \end{vmatrix}$$

and the characteristic equation is

$$\det|A - \lambda I| = -\lambda^3 + \lambda^2(-v_1 - v_2 - v_3 + \mu_1w_{11}g_1 + \mu_2w_{22}g_2 + \mu_3w_{33}g_3) + \lambda(g_1v_3\mu_1w_{11} +$$

$$+\mu_2w_{22}g_2v_3 + g_1g_2w_{21}\mu_1\mu_2w_{12} - g_1g_2w_{11}w_{22}\mu_1\mu_2 + g_1g_3w_{31}w_{13}\mu_1\mu_3 -$$

$$-g_1g_3w_{11}w_{33}\mu_1\mu_3 + g_2g_3w_{32}w_{23}\mu_2\mu_3 - g_2g_3w_{22}w_{33}\mu_2\mu_3 - v_1(v_2 + v_3 - g_2w_{22}\mu_2 - g_3w_{33}\mu_3) +$$

$$+v_2(-v_3 + g_1w_{11}\mu_1 + g_3w_{33}\mu_3)) + v_1(v_2(-v_3 + g_3w_{33}\mu_3) + g_2\mu_2(v_3w_{22} + g_3w_{32}w_{23}\mu_3 - g_3w_{22}w_{33}\mu_3)) +$$

$$+g_1\mu_3(v_2(v_3w_{11} + g_3(w_{31}w_{13} - w_{11}w_{33})\mu_3) + g_2\mu_2(v_3(w_{21}w_{12} - w_{11}w_{22}) +$$

$$+g_3(-w_{31}w_{22}w_{13}+w_{21}w_{32}\,w_{13}+w_{31}w_{12}w_{23}-w_{11}w_{32}\,w_{23}-w_{21}w_{12}w_{33}+w_{11}w_{22}\,w_{33})\mu_3))=0.$$

The characteristic equation can be rewritten as

$$-\lambda^3+A\lambda^2+B\lambda+C=0, \tag{14}$$

where

$$A=-(v_1+v_2+v_3)+g_1w_{11}\mu_1+g_2w_{22}\mu_2+g_3w_{33}\mu_3,$$

$$\begin{aligned}B\ &= \mu_1\mu_2w_{31}w_{13}g_1g_3-\mu_2\mu_3w_{32}w_{23}g_2g_3+\mu_1\mu_2w_{21}w_{12}g_1g_2\\ &-(\mu_2w_{22}g_2-v_2)(\mu_3w_{33}g_3-v_3)-(\mu_1w_{11}g_1-v_1)(\mu_3w_{33}g_3-v_3)\\ &-(\mu_1w_{11}g_1-v_1)(\mu_2w_{22}g_2-v_2),\end{aligned}$$

$$\begin{aligned}C\ &= (\mu_1w_{11}g_1-v_1)(\mu_2w_{22}g_2-v_2)(\mu_3w_{33}g_3-v_3)+\mu_1\mu_2\mu_3w_{21}w_{32}w_{23}g_1g_2g_3\\ &+\mu_1\mu_2\mu_3w_{31}w_{12}w_{23}g_1g_2g_3-\mu_1\mu_3w_{31}w_{13}g_1g_3(\mu_2w_{22}g_2-v_2)\\ &-\mu_2\mu_3w_{32}w_{23}g_2g_3(\mu_1w_{11}g_1-v_1)-\mu_1\mu_2w_{21}w_{12}g_1g_2(\mu_3w_{33}g_3-v_3).\end{aligned}$$

5.1.1 Facts

Proposition 5.1. *The vector field $(f_1(x_1,x_2,x_3),f_2(x_1,x_2,x_3),f_3(x_1,x_2,x_3))$, where f_1, f_2 and f_3 are the right sides of the equations in (9), is directed inward on the boundary of the domain $Q_3:=\{(x_1,x_2,x_3):0<x_1<\frac{1}{v_1},0<x_2<\frac{1}{v_2},0<x_3<\frac{1}{v_3}\}$.*

Proposition 5.2. *System (9) has at least one equilibrium (critical point). All equilibria are located in the open box $Q_3:=\{(x_1,x_2,x_3):0<x_1<\frac{1}{v_1},0<x_2<\frac{1}{v_2},0<x_3<\frac{1}{v_3}\}$.*

5.2 Critical points

The three-dimensional system has three eigenvalues. Two main possibilities exist: either the three eigenvalues are real or two of them are complex conjugates. A critical point is stable if all eigenvalues have negative real parts; it is unstable if at least one eigenvalue has positive real part.

- **Node**. All eigenvalues are real and have the same sign. The node is stable (unstable) when the eigenvalues are negative (positive) [97].
- **Saddle**. All eigenvalues are real and at least one of them is positive and at least one is negative. Saddles are always unstable [97].
- **Focus – Node**. It has one real eigenvalue and a pair of complex-conjugate eigenvalues, and all eigenvalues have real parts of the same sign. The critical point is stable (unstable) when the sign is negative (positive) [97].
- **Saddle – Focus**. Negative real eigenvalue and complex eigenvalues with positive real part (unstable focus), and positive real eigenvalue and complex eigenvalues with negative real part (stable focus). This type of critical point is unstable [46].

5.3 Chaos

Under chaos in ancient Greek mythology understood the pre-life confusion. Greek "chaos" is the infinite first everyday mass, which subsequently gave rise to all the existing. Physicists call this science - "nonlinear dynamics", mathematicians - "chaos theory", all the rest - "nonlinear science".

Chaos is a multifaceted phenomenon that is not easily classified or identified. There is no universally accepted definition for chaos, but the following characteristics are nearly always displayed by the solutions of chaotic systems [39].

Characteristics of chaos

- A characteristic of chaotic behavior is the existence of an attractor to which all sufficiently nearby solutions converge, given sufficient time [23].
- A typical characteristic of chaotic solutions is the geometric form of the attractors. The attractors typically are twisted and 'strange', meaning that they have fractional (fractal) dimension, although this is not necessarily the case [23].
- Sensitivity to initial conditions [39].

Definition 5.1. *A chaotic system is a deterministic system that exhibits irregular and unpredictable behaviour [47].*

Research on chaotic systems had a practical effect since Edward Norton Lorenz established chaos theory in 1963. Chaos should be expected to be a very common basic dynamical state in a variety of systems. Chaotic dynamics is very important in different fields such as robotics, economics, cryptography, chemistry, medicine (studying epilepsy to predict seizures, taking into account the initial state of the organism) and biology (in the study of uneven heart rate and an uneven number of diseases) [49].

Proposition 5.3. *In dynamical systems that include three or more equations, there may be even more unusual attractors, which are commonly called strange or chaotic attractors.*

Floris Takens (1940 - 2010) a Dutch mathematician known for contributions to the theory of differential equations, the theory of dynamical systems, chaos theory and fluid mechanics. Introduced the concept of a "strange attractor". He was the first to show how chaotic attractors could be learned by neural networks [7].

Proposition 5.4. *It is possible to find a chaotic attractor in differential systems presenting chaotic behaviour [55].*

Definition 5.2. *A strange attractor, (chaotic attractor, fractal attractor) is an attractor that exhibits sensitivity to initial conditions [39].*

Definition 5.3. *A fractal is an object that displays self-similarity under magnification and can be constructed using a simple motif (an image repeated on ever-reduced scales) [39].*

5.4 Lyapunov exponents

The Lyapunov exponents are an important tool for the characterization of an attractor of a finite-dimensional nonlinear dynamic system and their excessive sensitivity to initial conditions [19]. The Lyapunov exponent is an approach to detect chaos, and it is a measure of the speeds at which initially nearby trajectories of the system diverge [47].

Relationships between the Lyapunov exponents and the properties and types of attractors:

1. One-dimensional system. In this case only a stable fixed point can be an attractor. There exists one negative Lyapunov exponent (LE in short) denoted by $LE_1 = (-)$.
2. Two-dimensional system. In 2D systems, there are two types of attractors: stable fixed points and limit cycles. The corresponding LEs follow:
   - $(LE_1, LE_2) = (-, -)$ - stable point;
   - $(LE_1, LE_2) = (0, -)$ - stable limit cycle (one exponent is equal to zero).
3. Three-dimensional system. In 3D phase space, there exist four types of attractors: stable points, limit cycles, 2D tori and strange attractors. The following set of LEs characterizes possible dynamical situations to be met:
   - $(LE_1, LE_2, LE_3) = (-, -, -)$ - stable fixed point;
   - $(LE_1, LE_2, LE_3) = (0, -, -)$ - stable limit cycle;
   - $(LE_1, LE_2, LE_3) = (0, 0, -)$ - stable 2D tori;
   - $(LE_1, LE_2, LE_3) = (+, 0, -)$ - strange attractor.

5.4.1 Properties of Lyapunov exponents

1. The number of Lyapunov exponents is equal to the number of phase space dimensions, or the order of the system of differential equations. They are arranged in descending order [79].
2. The largest Lyapunov exponent of a stable system does not exceed zero [47].
3. A chaotic system has at least one positive Lyapunov exponent, and the more positive the largest Lyapunov exponent, the more unpredictable the system is [47].
4. To have a dissipative dynamical system, the values of all Lyapunov exponents should sum to a negative number [79].
5. A hyperchaotic system is defined as a chaotic system with at least two positive Lyapunov exponents. Combined with one null exponent and one negative exponent, the minimal dimension for a hyperchaotic system is four [86].

Proposition 5.5. *Only dissipative dynamical systems have attractors [46].*

In the thesis for Lyapunov exponents calculation the package "lce.m for Mathematica" was used [99]. Another Wolfram Mathematica program "Lynch-DSAM.nb" was also used to check the correctness of Lyapunov exponents calculation [39].

5.5 Examples

5.5.1 Periodic solutions

Example 1. Consider $\mu_1 = 5$, $\mu_2 = 15$, $\mu_3 = 5$, $v_1 = v_2 = v_3 = 1$ and $\theta_1 = 1.2$, $\theta_2 = 0.5$, $\theta_3 = -0.6$. The regulatory matrix of the system (9) is

$$W = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}. \tag{15}$$

The nullclines are depicted in Figure 4. There are exactly three critical points.

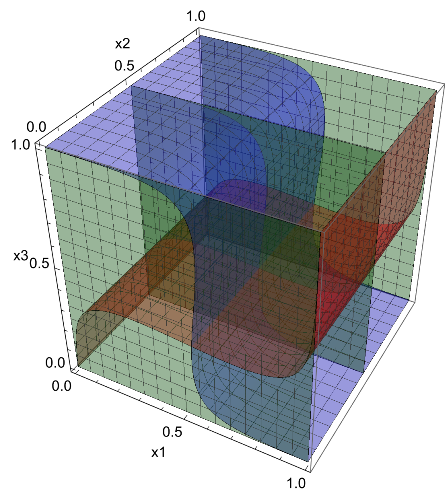

Figure 4: Nullclines x_1 - red, x_2 - green, x_3 - blue of the system (9) with the regulatory matrix (15).

The characteristic equation for critical point $(0.537; 0.001; 0.346)$ is

$$-\lambda^3 + A\lambda^2 + B\lambda + C = 0, \tag{16}$$

where $A = -0.616403$, $B = -5.28938$ and $C = -5.61417$.

Solving the equation we have $\lambda_1 = -0.99$, $\lambda_{2,3} = 0.188 \pm 2.371i$. The type of the critical point is unstable saddle-focus.

The characteristic equation for critical point $(0.537; 0.5; 0.346)$ is (16), where $A = 3.125$, $B = -6.693$ and $C = 15.569$.

Solving the equation we have $\lambda_1 = 2.75$, $\lambda_{2,3} = 0.187 \pm 2.371i$. The type of the critical point is unstable focus-node.

The characteristic equation for critical point $(0.537; 0.99; 0.346)$ is (16), where $A = -0.6164$, $B = -5.289$ and $C = -5.614$.

Solving the equation we have $\lambda_1 = -0.995$, $\lambda_{2,3} = 0.187 \pm 2.371i$. The type of the critical point is unstable saddle-focus.

There are three periodic solutions in Example 3. Periodic solutions are stable attractors. The solutions of the system (6) with the regulatory matrix (15) are depicted in Figure 5 and Figure 6.

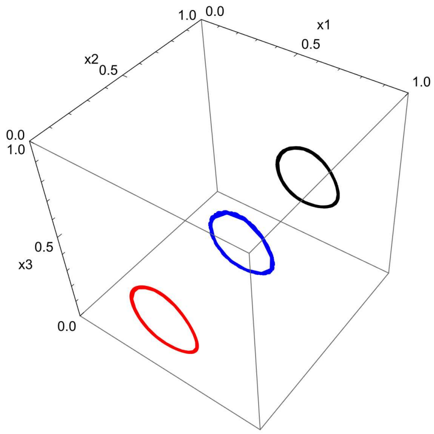

Figure 5: Example of two 3D limit cycles in the system (9) with the regulatory matrix (15).

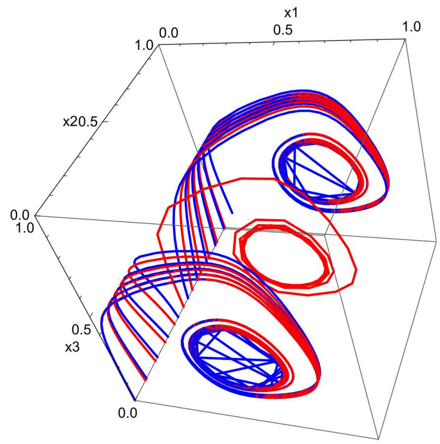

Figure 6: Three periodic solutions of the system (9) with the regulatory matrix (15).

5.5.2 Chaotic attractors

Consider

$$\mu_1 = \mu_2 = 7, \mu_3 = 13, v_1 = 0.65, v_2 = 0.42, v_3 = 0.1, \theta_1 = 0.5, \theta_2 = 0.3, \theta_3 = 0.7 \tag{17}$$

$$W = \begin{pmatrix} 0 & 1 & -5.65 \\ 1 & 0 & 0.135 \\ 1 & 0.02 & 0.03 \end{pmatrix}. \tag{18}$$

The initial conditions are

$$x_1(0) = 0.3;\ x_2(0) = 1.5;\ x_3(0) = 0.2. \tag{19}$$

The characteristic equation for critical point $(0.370457; 1.59272; 0.222436)$ is

$$-\lambda^3 + A\lambda^2 + B\lambda + C = 0,$$

where $A = -1.16152$, $B = -0.430187$ and $C = -0.688906$.

Solving the equation we have $\lambda_1 = -1.2558$, $\lambda_{2,3} = 0.0471391 \pm 0.739161i$. The type of the critical point is unstable saddle-focus. The system is a chaotic in the sense that solutions exhibit non-regular behavior. The self-excited chaotic attractor is depicted in Figure 7 .
The respective three-dimensional system was studied in [13], [14].

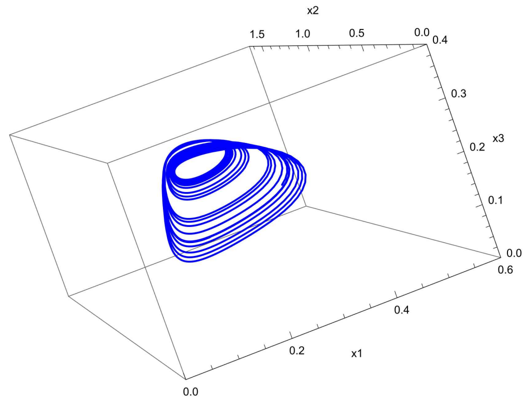

Figure 7: The self-excited chaotic attractor of the system (9) with the regulatory matrix (18).

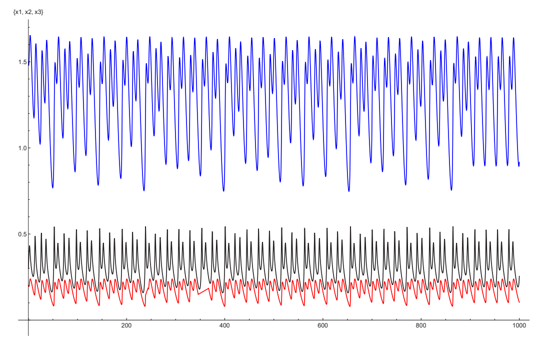

Figure 8: The graphs of $x_i(t), i = 1, 2, 3,$ of the system (9) with the regulatory matrix (18).

Now we change the parameter w_{23} (that is, the third element in the second row) in the regulatory matrix (18). The coordinates of a single critical point, values of the characteristic numbers for this point, are provided. Computations are performed using Wolfram Mathematica.

Table 1. Results of calculations for the system (9) with regulatory matrix (18), changing the parameter w_{23}.

w_{23}	x^*	y^*	z^*	Real λ	Complex λ $\mathbb{R}$ part	Complex λ im part
0.0	0.3651	1.4571	0.1989	-1.4269	0.1322	0.6634
0.05	0.3671	1.5057	0.2073	-1.3714	0.1047	0.6886
0.10	0.3691	1.5562	0.2161	-1.3069	0.0726	0.71698
0.12	0.3699	1.57699	0.2197	-1.2783	0.0583	0.7294
0.13	0.3703	1.5875	0.2215	-1.2634	0.0519	0.7359
0.132	0.3703	1.5895	0.2219	-1.2604	0.0494	0.7371
0.133	0.3704	1.5906	0.2221	-1.2589	0.0487	0.7378
0.134	0.3704	1.5917	0.2223	-1.2573	0.0479	0.7385
0.136	0.3705	1.5938	0.2226	-1.2589	0.0487	0.7378
0.137	0.3705	1.5948	0.2228	-1.2527	0.0456	0.7405
0.138	0.3706	1.5959	0.22299	-1.2512	0.0448	0.7412
0.139	0.3706	1.5969	0.2232	-1.2494	0.0441	0.7418
0.14	0.3706	1.59799	0.2234	-1.2481	0.0433	0.7425
0.145	0.3708	1.6033	0.2243	-1.2403	0.0394	0.7459
0.15	0.3710	1.6087	0.2252	-1.2324	0.0354	0.7493
0.16	0.3714	1.6192	0.2270	-1.2162	0.0274	0.7564
0.18	0.3721	1.6406	0.2308	-1.1826	0.0107	0.7711
0.19	0.3725	1.6514	0.2326	-1.1652	0.002	0.7787
0.20	0.3729	1.6622	0.2345	-1.1473	-0.0069	0.7867

Table 2. Lyapunov exponents for the system (9) with regulatory matrix (18), 8000 steps

w_{23}	LE_1	LE_2	LE_3	$LE_1 + LE_2 + LE_3$
0.0	0.00228824	-0.133556	-1.03537	-1.16664
0.13	0.00174998	-0.0409256	-1.12505	-1.16423
0.132	0.00241997	-0.0284958	-1.13784	-1.16392
0.133	0.00405175	0.00091658	-1.16866	-1.1637
0.134	0.0200966	0.000487689	-1.18412	-1.16354
0.135	0.0162669	0.000848416	-1.18055	-1.16343
0.136	0.00335708	-0.0065914	-1.16009	-1.16332
0.137	-0.000688284	-0.0214113	-1.14116	-1.16326
0.19	-0.00174416	-0.0102177	-1.14928	-1.16124
0.20	-0.00816703	-0.0105543	-1.14236	-1.16108
1	-0.373617	-0.37358	-0.409939	-1.15714

Calculations showed the following:

- if $0 \leq w_{23} < 0.132$, then the system (18) has a periodic solution;
- if $0.133 < w_{23} \leq 0.135$, then the system (18) has a chaotic solution;
- if $0.136 < w_{23} \leq 0.19$, then the system (18) has a periodic solution;
- if $w_{23} > 0.2$, then the system (18) has a stable fixed point.

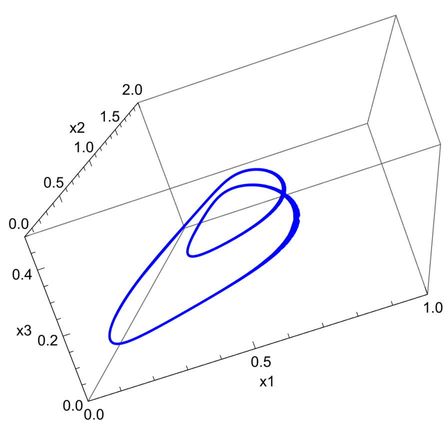

Figure 9: The periodic solution of the system (9) with the regulatory matrix (18), $w_{23} = 0.05$.

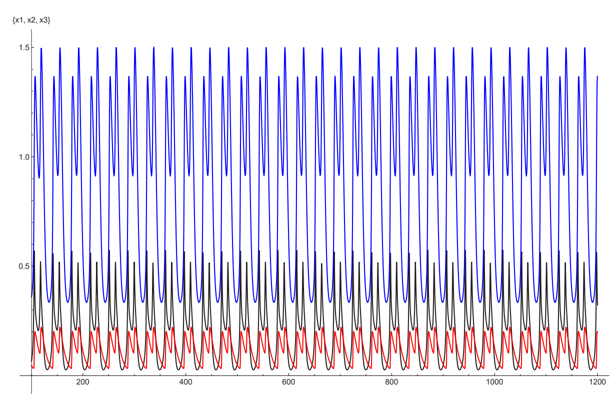

Figure 10: Solutions $(x_1(t), x_2(t), x_3(t))$ of the system (9) with the regulatory matrix (18), $w_{23} = 0.05$.

Now let change w_{32} values in the regulatory matrix (18).

Table 3. Results of calculations for the system (9) with regulatory matrix (18), changing the parameter w_{32}.

w_{23}	x^*	y^*	z^*	Real λ	Complex λ $\mathbb{R}$ part	Complex λ im part
0.0	0.4092	1.7387	0.2449	-1.036	-0.0623	0.8666
0.01	0.3892	1.6656	0.2337	-1.1554	-0.0029	0.7966
0.03	0.3530	1.5213	0.2114	-1.3366	0.0873	0.6912
0.04	0.3368	1.4523	0.2007	-1.3996	0.1186	0.6507

Table 4. Lyapunov exponents for the system (9) with regulatory matrix (18), 8000 steps

w_{23}	LE_1	LE_2	LE_3	$LE_1 + LE_2 + LE_3$
0.0	-0.063377	-0.0643758	-0.406599	-1.16067
0.01	-0.00456357	-0.00807802	-1.14848	-1.16113
0.02	0.0162669	0.000848416	-1.18055	-1.16343
0.03	0.0015434	-0.186553	-0.980366	-1.16538
0.04	0.00381232	-0.0985729	-1.07168	-1.16644

Calculations showed the following:

- if $0 \leq w_{32} \leq 0.01$, then the system (18) has a stable fixed point;
- if $w_{32} = 0.02$, then the system (18) has a chaotic solution;
- if $0.03 \leq w_{32} \leq 0.04$, then the system (18) has a periodic solution.

From calculations we see that small changes in parameter values change the behavior of the system.

6 Four-dimensional (4D) systems

Consider four-dimensional system

$$\begin{cases} \dfrac{dx_1}{dt} = \dfrac{1}{1 + e^{-\mu_1(w_{11}x_1 + w_{12}x_2 + w_{13}x_3 + w_{14}x_4 - \theta_1)}} - v_1 x_1, \\ \dfrac{dx_2}{dt} = \dfrac{1}{1 + e^{-\mu_2(w_{21}x_1 + w_{22}x_2 + w_{23}x_3 + w_{24}x_4 - \theta_2)}} - v_2 x_2, \\ \dfrac{dx_3}{dt} = \dfrac{1}{1 + e^{-\mu_3(w_{31}x_1 + w_{32}x_2 + w_{33}x_3 + w_{34}x_4 - \theta_2)}} - v_3 x_3, \\ \dfrac{dx_4}{dt} = \dfrac{1}{1 + e^{-\mu_4(w_{41}x_1 + w_{42}x_2 + w_{43}x_3 + w_{44}x_4 - \theta_4)}} - v_4 x_4. \end{cases} \tag{20}$$

The nullclines are given by

$$\begin{cases} v_1x_1 = \dfrac{1}{1+e^{-\mu_1(w_{11}x_1+w_{12}x_2+w_{13}x_3+w_{14}x_4-\theta_1)}}, \\ v_2x_2 = \dfrac{1}{1+e^{-\mu_2(w_{21}x_1+w_{22}x_2+w_{23}x_3+w_{24}x_4-\theta_2)}}, \\ v_3x_3 = \dfrac{1}{1+e^{-\mu_3(w_{31}x_1+w_{32}x_2+w_{33}x_3+w_{34}x_4-\theta_2)}}, \\ v_4x_4 = \dfrac{1}{1+e^{-\mu_4(w_{41}x_1+w_{42}x_2+w_{43}x_3+w_{44}x_4-\theta_4)}}. \end{cases} \tag{21}$$

Critical points are solutions of the system (21).

6.1 Linearized system

The linearized system for critical point $(x_1^*, x_2^*, x_3^*, x_4^*)$ is

$$\begin{cases} u_1' = -v_1u_1 + \mu_1w_{11}g_1u_1 + \mu_1w_{12}g_1u_2 + \mu_1w_{13}g_1u_3 + \mu_1w_{14}g_1u_4, \\ u_2' = -v_2u_2 + \mu_2w_{21}g_2u_1 + \mu_2w_{22}g_2u_2 + \mu_2w_{23}g_2u_3 + \mu_2w_{24}g_2u_4, \\ u_3' = -v_3u_3 + \mu_3w_{31}g_3u_1 + \mu_3w_{32}g_3u_2 + \mu_3w_{33}g_3u_3 + \mu_3w_{34}g_3u_4, \\ u_4' = -v_4u_4 + \mu_4w_{41}g_4u_1 + \mu_4w_{42}g_4u_2 + \mu_4w_{34}g_4u_3 + \mu_4w_{44}g_4u_4, \end{cases}$$

where

$$g_1 = \frac{e^{-\mu_1(w_{11}x_1^*+w_{12}x_2^*+w_{13}x_3^*+w_{14}x_4^*-\theta_1)}}{[1+e^{-\mu_1(w_{11}x_1^*+w_{12}x_2^*+w_{13}x_3^*+w_{14}x_4^*-\theta_1)}]^2},$$

$$g_2 = \frac{e^{-\mu_2(w_{21}x_1^*+w_{22}x_2^*+w_{23}x_3^*+w_{24}x_4^*-\theta_2)}}{[1+e^{-\mu_2(w_{21}x_1^*+w_{22}x_2^*+w_{23}x_3^*+w_{24}x_4^*-\theta_2)}]^2},$$

$$g_3 = \frac{e^{-\mu_3(w_{31}x_1^*+w_{32}x_2^*+w_{33}x_3^*+w_{34}x_4^*-\theta_3)}}{[1+e^{-\mu_3(w_{31}x_1^*+w_{32}x_2^*+w_{33}x_3^*+w_{34}x_4^*-\theta_3)}]^2},$$

$$g_4 = \frac{e^{-\mu_4(w_{41}x_1^*+w_{42}x_2^*+w_{43}x_3^*+w_{44}x_4^*-\theta_4)}}{[1+e^{-\mu_4(w_{41}x_1^*+w_{42}x_2^*+w_{43}x_3^*+w_{44}x_4^*-\theta_4)}]^2}.$$

The characteristic equation is

$$\lambda^4 + A\lambda^3 + B\lambda^2 + M\lambda + L = 0, \tag{22}$$

where

$$A = (v_1 + v_2 + v_3 + v_4) - g_1w_{11}\mu_1 - g_2w_{22}\mu_2 - g_3w_{33}\mu_3 - g_4\mu_4w_{44},$$

$$\begin{aligned} B = {} & v_3v_4 - g_1v_3w_{11}\mu_1 - g_1v_4w_{11}\mu_1 - g_2v_3w_{22}\mu_2 - g_2v_4w_{22}\mu_2 - g_1g_2w_{21}w_{12}\mu_1\mu_2 \\ & + g_1g_2w_{11}w_{22}\mu_1\mu_2 - g_3v_4w_{33}\mu_3 - g_1g_3w_{31}w_{13}\mu_1\mu_3 + g_1g_3w_{11}w_{33}\mu_1\mu_3 \\ & - g_2g_3w_{32}w_{23}\mu_2\mu_3 + g_2g_3w_{22}w_{33}\mu_2\mu_3 - g_1g_4w_{41}\mu_1\mu_4w_{14} - g_2g_4w_{42}\mu_2\mu_4w_{24} \\ & - g_3g_4w_{43}\mu_3\mu_4w_{34} - g_4v_3\mu_4w_{44} + g_1g_4w_{11}\mu_1\mu_4w_{44} + g_2g_4w_{22}\mu_2\mu_4w_{44} + g_3g_4w_{33}\mu_3\mu_4w_{44} \\ & + v_2(v_3+v_4-g_1w_{11}\mu_1-g_3w_{33}\mu_3-g_4\mu_4w_{44}) + v_1(v_2+v_3+v_4-g_2w_{22}\mu_2-g_3w_{33}\mu_3-g_4\mu_4w_{44}), \end{aligned}$$

$$
\begin{aligned}
M = &- g_1v_3v_4w_{11}\mu_1 - g_2v_3v_4w_{22}\mu_2 - g_1g_2v_3w_{21}w_{12}\mu_1\mu_2 - g_1g_2v_4w_{21}w_{12}\mu_1\mu_2 \\
&+ g_1g_2v_3w_{11}w_{22}\mu_1\mu_2 + g_1g_2v_4w_{11}w_{22}\mu_1\mu_2 - g_1g_3v_4w_{31}w_{13}\mu_1\mu_3 \\
&+ g_1g_3v_4w_{11}w_{33}\mu_1\mu_3 - g_2g_3v_4w_{32}w_{23}\mu_2\mu_3 + g_2g_3v_4w_{22}w_{33}\mu_2\mu_3 \\
&+ g_1g_2g_3w_{31}w_{22}w_{13}\mu_1\mu_2\mu_3 - g_1g_2g_3w_{21}w_{32}w_{13}\mu_1\mu_2\mu_3 - g_1g_2g_3w_{31}w_{12}w_{23}\mu_1\mu_2\mu_3 \\
&+ g_1g_2g_3w_{11}w_{32}w_{23}\mu_1\mu_2\mu_3 + g_1g_2g_3w_{21}w_{12}w_{33}\mu_1\mu_2\mu_3 - g_1g_2g_3w_{11}w_{22}w_{33}\mu_1\mu_2\mu_3 \\
&- g_1g_4v_3w_{41}\mu_1\mu_4w_{14} + g_1g_2g_4w_{41}w_{22}\mu_1\mu_2\mu_4w_{14} - g_1g_2g_4w_{21}w_{42}\mu_1\mu_2\mu_4w_{14} \\
&+ g_1g_3g_4w_{41}w_{33}\mu_1\mu_3\mu_4w_{14} - g_1g_3g_4w_{31}w_{43}\mu_1\mu_3\mu_4w_{14} - g_2g_4v_3w_{42}\mu_2\mu_4w_{24} \\
&- g_1g_2g_4w_{41}w_{12}\mu_1\mu_2\mu_4w_{24} + g_1g_2g_4w_{11}w_{42}\mu_1\mu_2\mu_4w_{24} + g_2g_3g_4w_{42}w_{33}\mu_2\mu_3\mu_4w_{24} \\
&- g_2g_3g_4w_{32}w_{43}\mu_2\mu_3\mu_4w_{24} - g_1g_3g_4w_{41}w_{13}\mu_1\mu_3\mu_4w_{34} + g_1g_3g_4w_{11}w_{43}\mu_1\mu_3\mu_4w_{34} \\
&- g_2g_3g_4w_{42}w_{23}\mu_2\mu_3\mu_4w_{34} + g_2g_3g_4w_{22}w_{43}\mu_2\mu_3\mu_4w_{34} + g_1g_4v_3w_{11}\mu_1\mu_4w_{44} \\
&+ g_2g_4v_3w_{22}\mu_2\mu_4w_{44} + g_1g_2g_4w_{21}w_{12}\mu_1\mu_2\mu_4w_{44} - g_1g_2g_4w_{11}w_{22}\mu_1\mu_2\mu_4w_{44} \\
&+ g_1g_3g_4w_{31}w_{13}\mu_1\mu_3\mu_4w_{44} - g_1g_3g_4w_{11}w_{33}\mu_1\mu_3\mu_4w_{44} + g_2g_3g_4w_{32}w_{23}\mu_2\mu_3\mu_4w_{44} \\
&- g_2g_3g_4w_{22}w_{33}\mu_2\mu_3\mu_4w_{44} + v_1(v_3v_4 - g_2v_3w_{22}\mu_2 - g_2v_4w_{22}\mu_2 - g_3v_4w_{33}\mu_3 \\
&- g_2g_3w_{32}w_{23}\mu_2\mu_3 + g_2g_3w_{22}w_{33}\mu_2\mu_3 - g_2g_4w_{42}\mu_2\mu_4w_{24} \\
&- g_3g_4w_{43}\mu_3\mu_4w_{34} - g_4v_3\mu_4w_{44} + g_2g_4w_{22}\mu_2\mu_4w_{44} \\
&+ g_3g_4w_{33}\mu_3\mu_4w_{44} + v_2(v_3 + v_4 - g_3w_{33}\mu_3 - g_4\mu_4w_{44})) \\
&+ v_2(v_3(v_4 - g_1w_{11}\mu_1 - g_4\mu_4w_{44}) - g_1\mu_1(v_4w_{11} + g_3w_{31}w_{13}\mu_3 - g_3w_{11}w_{33}\mu_3 + g_4w_{41}\mu_4w_{14} \\
&- g_4w_{11}\mu_4w_{44}) - g_3\mu_3(v_4w_{33} + g_4w_{43}\mu_4w_{34} - g_4w_{33}\mu_4w_{44})),
\end{aligned}
$$

$$
\begin{aligned}
L = &\, v_1(v_2(v_3(v_4 - g_4\mu_4w_{44}) - g_3\mu_3(v_4w_{33} + g_4w_{43}\mu_4w_{34} - g_4w_{33}\mu_4w_{44})) \\
&- g_2\mu_2(v_3(v_4w_{22} + g_4\mu_4(w_{42}w_{24} - w_{22}w_{44})) + g_3\mu_3(v_4(w_{32}w_{23} - w_{22}w_{33}) \\
&+ g_4\mu_4(-w_{42}w_{33}w_{24} + w_{32}w_{43}w_{24} + w_{42}w_{23}w_{34} - w_{22}w_{43}w_{34} - w_{32}w_{23}w_{44} + w_{22}w_{33}w_{44})))) \\
&- g_1\mu_1(v_2(v_3(v_4w_{11} + g_4\mu_4(w_{41}w_{14} - w_{11}w_{44})) + g_3\mu_3(v_4(w_{31}w_{13} - w_{11}w_{33}) \\
&+ g_4\mu_4(-w_{41}w_{33}w_{14} + w_{31}w_{43}w_{14} + w_{41}w_{13}w_{34} - w_{11}w_{43}w_{34} - w_{31}w_{13}w_{44} + w_{11}w_{33}\ w_{44}))) \\
&+ g_2\mu_2(v_3(v_4(w_{21}w_{12} - w_{11}w_{22}) \\
&+ g_4\mu_4(-w_{41}w_{22}w_{14} + w_{21}w_{42}\ w_{14} + w_{41}w_{12}w_{24} - w_{11}w_{42}w_{24} - \ w_{21}w_{12}w_{44} + w_{11}w_{22}w_{44})) \\
&+ g_3\mu_3(v_4(-w_{31}w_{22}w_{13} + w_{21}\ w_{32}w_{13} + w_{31}w_{12}w_{23} - w_{11}\ w_{32}w_{23} - w_{21}w_{12}w_{33} + w_{11}\ w_{22}w_{33}) \\
&+ g_4\mu_4(-w_{21}w_{42}w_{33}w_{14} + w_{21}w_{32}w_{43}w_{14} + w_{11}w_{42}w_{33}w_{24} - w_{11}w_{32}w_{43}w_{24} \\
&+ w_{21}w_{42}w_{13}w_{34} - w_{11}w_{42}w_{23}w_{34} - w_{21}w_{12}w_{43}w_{34} + w_{11}w_{22}w_{43}w_{34} \\
&+ w_{41}(-w_{32}w_{23}w_{14} + w_{22}w_{33}w_{14} + w_{32}w_{13}w_{24} - w_{12}w_{33}w_{24} - w_{22}w_{13}w_{34} + w_{12}w_{23}w_{34}) \\
&- w_{21}w_{32}w_{13}w_{44} + w_{11}w_{32}w_{23}w_{44} + w_{21}\ w_{12}w_{33}w_{44} - w_{11}w_{22}w_{33}w_{44} \\
&+ w_{31}(w_{42}w_{23}w_{14} - w_{22}w_{43}w_{14} - w_{42}w_{13}w_{24} + w_{12}w_{43}w_{24} + w_{22}w_{13}w_{44} - w_{12}w_{23}w_{44}))))).
\end{aligned}
$$

6.2 Critical points

The four-dimensional system has 4 eigenvalues.

- **4D node**. All eigenvalues are real and have the same sign. The node is stable (unstable) when the eigenvalues are negative (positive).
- **4D star**. All eigenvalues are equal. The 4D star is stable (unstable) when the eigenvalues are negative (positive).

- **Saddle**. All eigenvalues are real and at least one of them is positive and at least one is negative. Saddles are always unstable.
- **Focus – Node**. It has two real eigenvalues and a pair of complex-conjugate eigenvalues, and all eigenvalues have real parts of the same sign. The critical point is stable (unstable) when the sign is negative (positive).
- **Node – Focus**. It has two real negative eigenvalues and a pair of complex-conjugate eigenvalues with positive real part. The critical point is unstable.
- **Saddle – Focus**. Two real eigenvalues have different signs and complex-conjugate eigenvalues with positive or negative real part. The critical point is unstable.
- **Focus – Focus**. Two pairs of complex-conjugate eigenvalues. The critical point is stable when the signs of real parts are negative. The critical point is unstable when there is at least one positive real part.

6.3 Lyapunov exponents

Relationships between the Lyapunov exponents and the properties and types of attractors:

- $(LE_1, LE_2, LE_3, LE_4) = (-,-,-,-)$ - stable fixed point;
- $(LE_1, LE_2, LE_3, LE_4) = (0,-,-,-)$ - periodic solutions (limit cycles);
- $(LE_1, LE_2, LE_3, LE_4) = (0,0,-,-)$ - quasiperiodic solution;
- $(LE_1, LE_2, LE_3, LE_4) = (+,0,-,-)$ - strange attractor;
- $(LE_1, LE_2, LE_3, LE_4) = (+,+,0,-)$ - hyperchaotic attractor [40].

6.4 Examples

Example 1. Consider the system (20) with the regulatory matrix,

$$W = \begin{pmatrix} k_1 & 2 & 0 & 0 \\ -2 & k_1 & 0 & 0 \\ 0 & 0 & k_2 & 2 \\ 0 & 0 & -2 & k_2 \end{pmatrix}, \tag{23}$$

where $k_1 = 0.5$, $k_2 = 1.815$ and $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 10$, $v_1 = v_2 = v_3 = v_4 = 1$, $\theta_1 = 1.2$, $\theta_2 = -0.7$, $\theta_3 = 1.8$, $\theta_4 = -0.28$

The initial conditions are

$$x_1(0) = 0.5;\ x_2(0) = 0.32;\ x_3(0) = 0.4;\ x_4(0) = 0.39.$$

This system consists of two independent two-dimensional systems. There is exactly one critical point. The standard linearization analysis provides the characteristic numbers $\lambda_{1,2} = 0.2469 \pm 4.9875i$; $\lambda_{3,4} = 3.4667 \pm 4.9215i$. The type of the critical point is an unstable focus-focus.

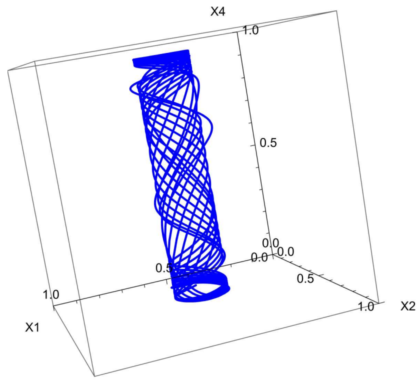

Figure 11: The projection of 4D trajectories to 3D subspace $(x_1(t), x_2(t), x_4(t))$.

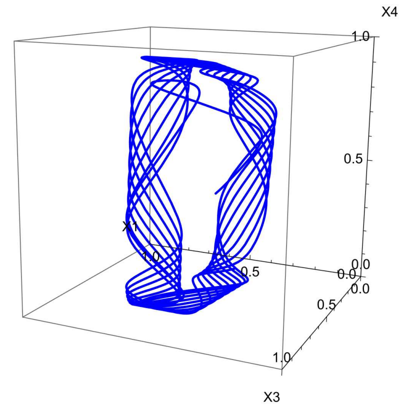

Figure 12: The projection of 4D trajectories to 3D subspace $(x_1(t), x_3(t), x_4(t))$.

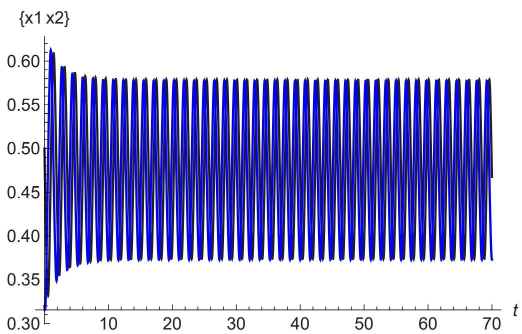

Figure 13: The graphs of periodic solutions $(x_1(t), x_2(t))$ of the system (20) with the regulatory matrix (23), $k_1 = 0.5$ and $k_2 = 1.815$.

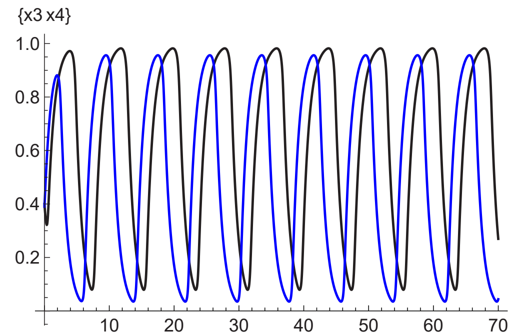

Figure 14: The graphs of periodic solutions $(x_3(t), x_4(t))$ of the system (20) with the regulatory matrix (23), $k_1 = 0.5$ and $k_2 = 1.815$.

Let us change two elements at the right upper (w_{14}) and left lower (w_{41}) corners. Let $w_{41} = 0.1$ and (w_{14}) values are considered in Table 5.

Table 5. Results of calculations for the system (20) with regulatory matrix (23) $k_1 = 0.5$ and $k_2 = 1.815$, changing the parameter w_{14}.

w_{14}	$\lambda_{1,2}$	$\lambda_{3,4}$	Lyapunov exponents
-1.2	$0.189 \pm 4.49i$	$3.374 \pm 4.912i$	(0; -0.48; -0.89; -0.96)
-1.1	$0.206 \pm 4.586i$	$3.379 \pm 4.908i$	(0; -0.70; -0.70; -0.87)
-1	$0.220 \pm 4.671i$	$3.384 \pm 4.905i$	(0.05; 0; -0.88; -0.98)
-0.9	$0.232 \pm 4.745i$	$3.389 \pm 4.902i$	(0; -0.27; -0.29; -0.89)
-0.8	$0.242 \pm 4.808i$	$3.394 \pm 4.899i$	(0; -0.05; -0.58; -0.88)
-0.7	$0.250 \pm 4.862i$	$3.399 \pm 4.897i$	(0.03; 0; -0.26; -0.89)
-0.6	$0.256 \pm 4.906i$	$3.405 \pm 4.896i$	(0; -0.20; -0.20; -0.89)
-0.5	$0.260 \pm 4.941i$	$3.410 \pm 4.894i$	(0; -0.09; -0.35; -0.89)
-0.4	$0.261 \pm 4.968i$	$3.415 \pm 4.893i$	(0; -0.13; -0.33; -0.89)

Calculations showed the following:

- if $-1.2 \leq w_{14} < -1$, then the system (20) with the regulatory matrix (23) has a periodic solution;
- if $w_{14} = -1$, then the system (20) with the regulatory matrix (23) is chaotic;
- if $-0.9 \leq w_{14} < -0.7$, then the system (20) with the regulatory matrix (23) has a periodic solution;
- if $w_{14} = -0.7$, then the system (20) with the regulatory matrix (23) is chaotic;
- if $-0.6 \leq w_{14} \leq -0.4$, then the system (20) with the regulatory matrix (23) has a periodic solution.

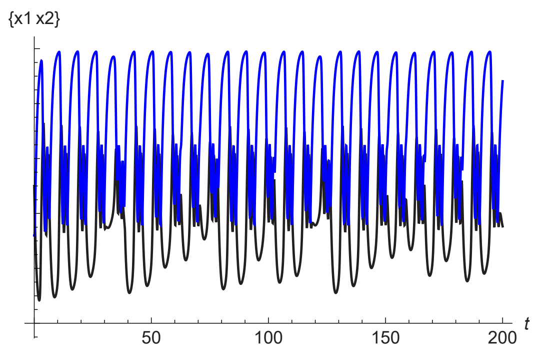

Figure 15: The graphs of solutions $(x_1(t), x_2(t))$ of the system (20) with the regulatory matrix (23), $k_1 = 0.5$ and $k_2 = 1.815$, $w_{14} = -1$.

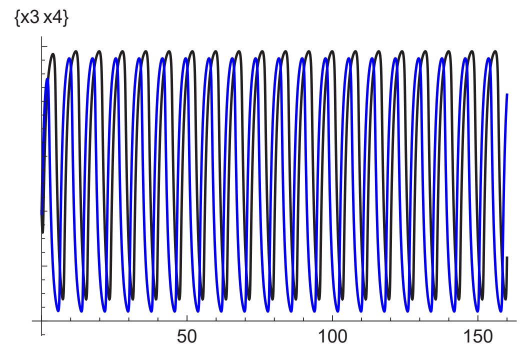

Figure 16: The graphs of solutions $(x_3(t), x_4(t))$ of the system (20) with the regulatory matrix (23), $k_1 = 0.5$ and $k_2 = 1.815$, $w_{14} = -1$.

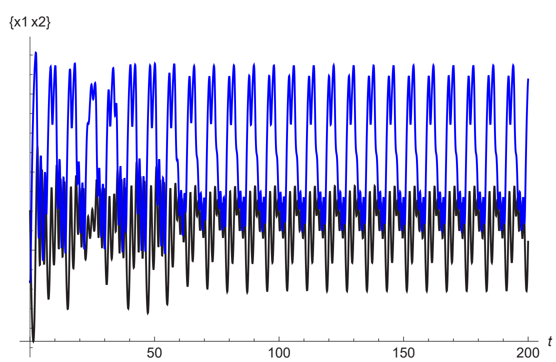

Figure 17: The graphs of solutions $(x_1(t), x_2(t))$ of the system (20) with the regulatory matrix (23), $k_1 = 0.5$ and $k_2 = 1.815$, $w_{14} = -0.7$.

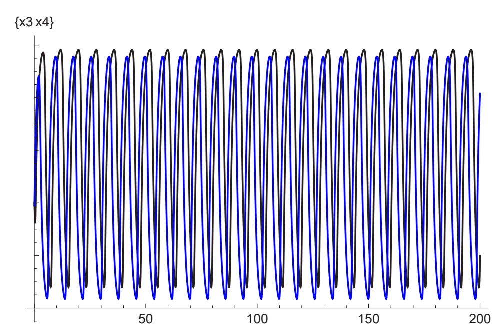

Figure 18: The graphs of solutions $(x_3(t), x_4(t))$ of the system (20) with the regulatory matrix (23), $k_1 = 0.5$ and $k_2 = 1.815$, $w_{14} = -0.7$.

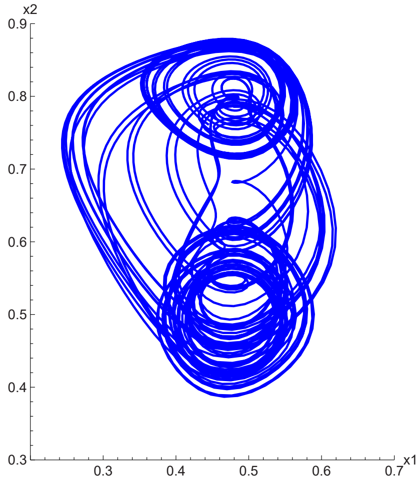

Figure 19: The projection of 4D trajectories to 2D subspace $(x_1(t), x_2(t)), w_{14} = -0.7$.

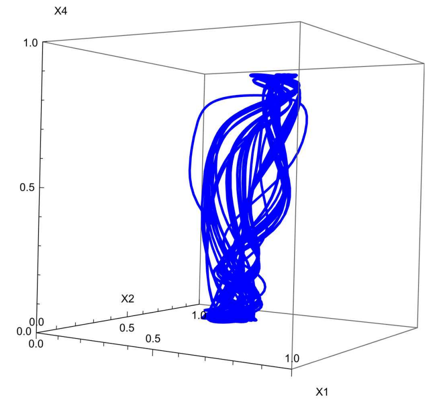

Figure 20: The projection of 4D trajectories to 3D subspace $(x_1(t), x_2(t), x_4(t)), w_{14} = -0.7$.

The dynamics of Lyapunov exponents are shown in Figure 21 and Figure 22.

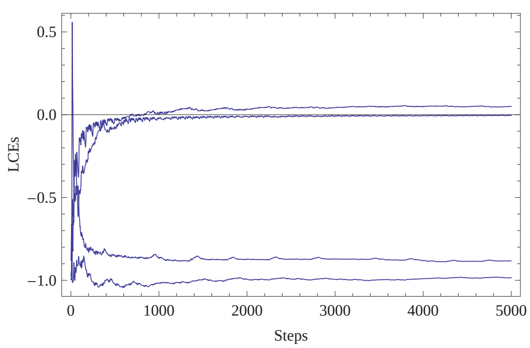

Figure 21: The dynamics of Lyapunov exponents of the system (20) with the regulatory matrix (23), $k_1 = 0.5$ and $k_2 = 1.815$, $w_{14} = -1$.

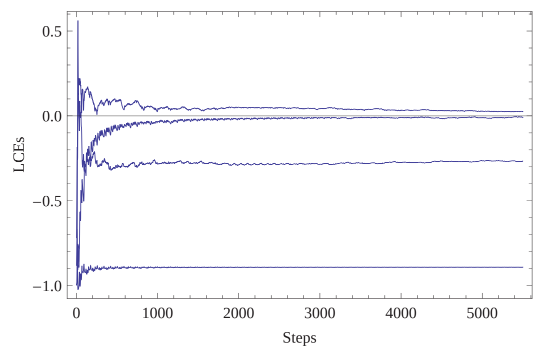

Figure 22: The dynamics of Lyapunov exponents of the system (20) with the regulatory matrix (23), $k_1 = 0.5$ and $k_2 = 1.815$, $w_{14} = -0.7$.

Example 2. The regulatory matrix is

$$W = \begin{pmatrix} 0.8 & 2 & -0.8 & 0.5 \\ -2 & 0.3 & 0.4 & -0.7 \\ -0.5 & 0.2 & 1.8 & 2 \\ 0.8 & -0.7 & -2 & 1.8 \end{pmatrix} \tag{24}$$

and the parameters $v_1 = v_2 = v_3 = v_4 = 1$, $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 10$ and θ_i, where

$i = 1, 2, 3, 4$ is calculated as

$$\begin{cases} \theta_1 = \dfrac{w_{11} + w_{12} + w_{13} + w_{14}}{2}, \\ \\ \theta_2 = \dfrac{w_{21} + w_{22} + w_{23} + w_{24}}{2}, \\ \\ \theta_3 = \dfrac{w_{31} + w_{32} + w_{33} + w_{34}}{2}, \\ \\ \theta_4 = \dfrac{w_{41} + w_{42} + w_{43} + w_{44}}{2}. \end{cases}$$

$\theta_1 = 1.25$, $\theta_2 = -1$, $\theta_3 = 1.75$, $\theta_4 = -0.05$.

The initial conditions are

$$x_1(0) = 0.4;\ x_2(0) = 0.6;\ x_3(0) = 0.39;\ x_4(0) = 0.38. \tag{25}$$

The critical point is $(0.5; 0.5; 0.5; 0.5)$. The standard linearization analysis provides the characteristic numbers $\lambda_{1,2} = -0.44 \pm 4.603i$ and $\lambda_{3,4} = 4.33 \pm 5.135i$. The type of the critical point is an unstable focus-focus.

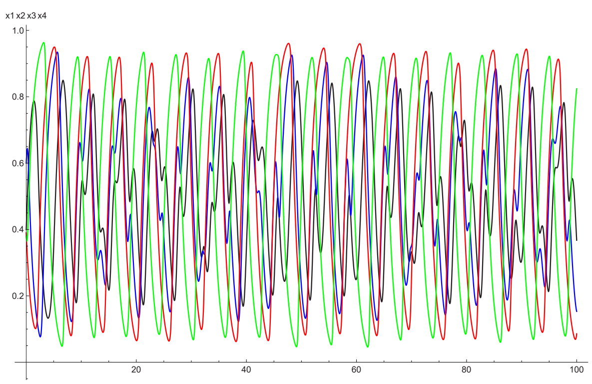

Figure 23: The graphs of solutions $(x_1(t), x_2(t), x_3(t), x_4(t))$ of the system (20) with the regulatory matrix (24).

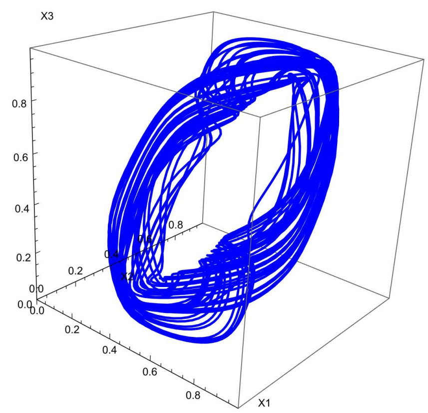

Figure 24: The projection of 4D trajectories to 3D subspace (x_1, x_2, x_3).

The dynamics of Lyapunov exponents are shown in Figure 25.

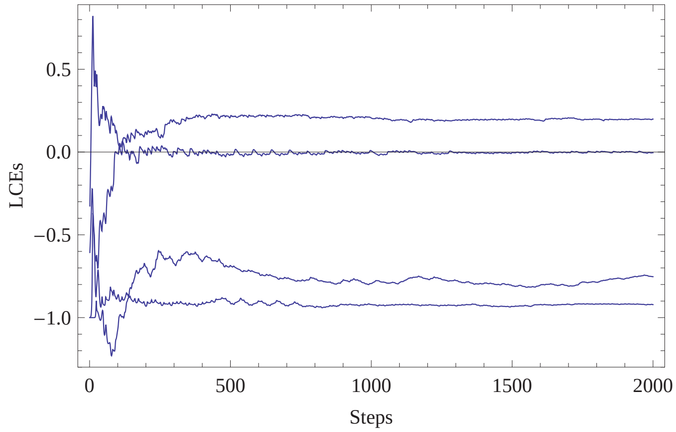

Figure 25: $LE_1 = 0.20, LE_2 = 0, LE_3 = -0.75, LE_4 = -0.92$

$LE_1, LE_2, LE_3, LE_4 = (+, 0, -, -)$ is a the self-excited chaotic attractor. The behavior of the system (20) with regulatory matrix (24) and initial conditions (25) is chaotic.

7 Five-dimensional (5D) systems

The system of ODE consisting of five equations is

$$\begin{cases} \dfrac{dx_1}{dt} = f_1(w_{11}x_1 + \ldots + w_{15}x_5) - v_1 x_1, \\ \dfrac{dx_2}{dt} = f_2(w_{21}x_1 + \ldots + w_{25}x_5) - v_2 x_2, \\ \qquad\qquad \ldots\ldots\ldots \\ \dfrac{dx_5}{dt} = f_5(w_{51}x_1 + \ldots w_{55}x_5) - v_5 x_5. \end{cases} \tag{26}$$

7.1 Examples

Example 1. Consider the five-dimensional system (26). Let the regulatory matrix be

$$W = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{27}$$

and $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = 5$, $v_1 = v_2 = v_3 = v_4 = v_5 = 1$, $\theta_1 = 1.5$, $\theta_2 = \theta_3 = 1$ and $\theta_4 = \theta_5 = 0.5$.
This system consists of one three-dimensional system and one two-dimensional system.

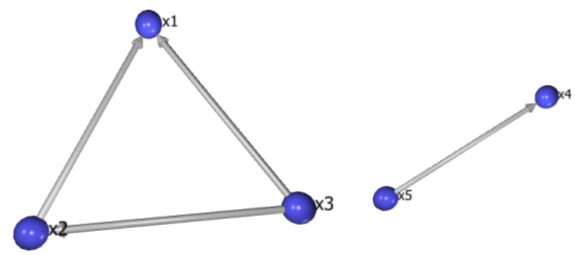

Figure 26: The graph, corresponding to the case of the regulatory matrix (27).

This system is uncoupled and has one critical point $(0.5, 0.5, 0.5, 0.5, 0.5)$. The solution of the system (26) with the regulatory matrix (27) is stable.

Example 2. Consider the five-dimensional system (26). Let the regulatory matrix be

$$W = \begin{pmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 2 \\ 0 & 0 & 0 & -2 & 0.5 \end{pmatrix} \tag{28}$$

and $\mu_1 = \mu_3 = 0.5$, $\mu_2 = 15$, $\mu_4 = \mu_5 = 10$, $v_1 = v_2 = v_3 = v_4 = v_5 = 1$, $\theta_1 = 1.2$, $\theta_2 = 0.5$, $\theta_3 = -0.6$, $\theta_4 = 1.2$, $\theta_5 = -0.7$.
This system consists of one three-dimensional system, which has a periodic solution depicted in Figure 5, and one two-dimensional system, which has a periodic solution depicted in Figure 1. This system is uncoupled and has three critical points. The solution of the system (26) with regulatory matrix (28) is periodic.

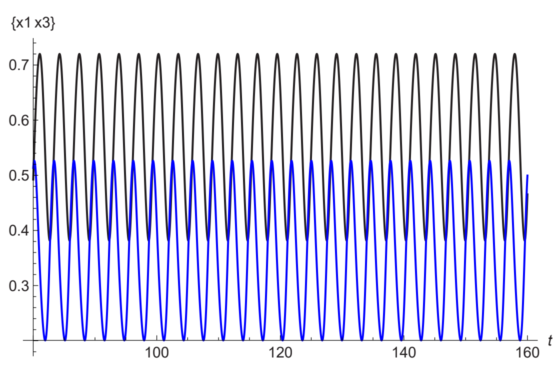

Figure 27: The graphs of solutions $x_i(t)$, $i = 1, 3$ of the system (26) with the regulatory matrix (28).

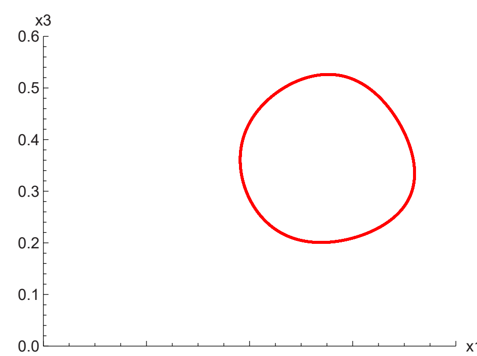

Figure 28: The projection of 5D trajectories to 2D subspace (x_1, x_3).

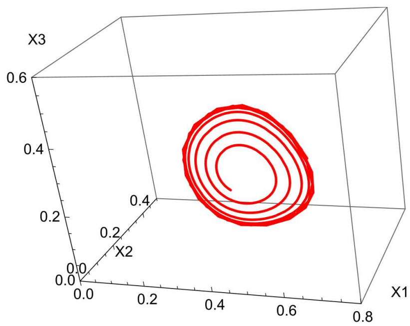

Figure 29: The projection of 5D trajectories to 3D subspace (x_1, x_2, x_3).

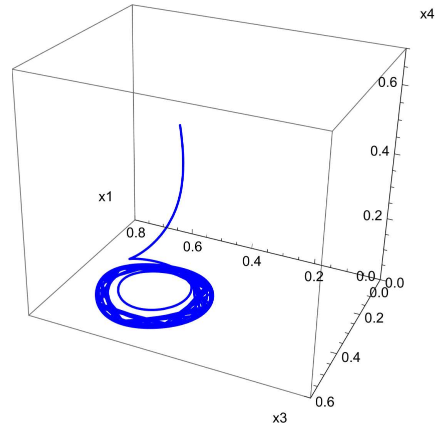

Figure 30: The projection of 5D trajectories to 3D subspace (x_1, x_3, x_4).

8 Six-dimensional (6D) systems

The system of ODE consisting of six equations is

$$\begin{cases} \dfrac{dx_1}{dt} = f_1(w_{11}x_1 + \ldots + w_{16}x_6) - v_1 x_1, \\[2ex] \dfrac{dx_2}{dt} = f_2(w_{21}x_1 + \ldots + w_{26}x_6) - v_2 x_2, \\[2ex] \qquad\qquad \cdots\cdots\cdots \\[2ex] \dfrac{dx_6}{dt} = f_6(w_{61}x_1 + \ldots w_{66}x_6) - v_6 x_6. \end{cases} \tag{29}$$

Similar systems of dimensionality two, three, four and of arbitrary dimensionality [58],[63] appear in various contexts describing neuronal networks [14],[13], genetic networks [89], telecommunications networks [29] and more. This type models can reflect an evolution in time t of a network. Networks management and control are possible by changing system parameters [70], [4].

8.1 Examples

Our intent now is to create a six-dimensional attractor from three-dimensional ones.

Example 1. Consider the six-dimensional system (29) with the regulatory matrix

$$W = \begin{pmatrix} k_1 & 0 & -1 & 0 & 0 & 0 \\ -1 & k_1 & 0 & 0 & 0 & 0 \\ 0 & -1 & k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_2 & 0 & -1 \\ 0 & 0 & 0 & -1 & k_2 & 0 \\ 0 & 0 & 0 & 0 & -1 & k_2 \end{pmatrix}, \tag{30}$$

where $k_1 = k_2 = 1$, $\mu_i = 5$, $\theta_i = \frac{k-1}{2}$.

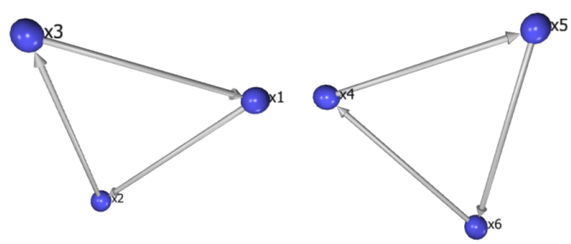

Figure 31: The graph, corresponding to the case of the regulatory matrix (30).

The initial conditions are

$$x_1(0) = 0.046;\ x_2(0) = 0.8;\ x_3(0) = 0.3;\ x_4(0) = 0.7;\ x_5(0) = 0.8;\ x_6(0) = 0.2.$$

The $6D$ system has an attractor in the form of a periodic solution generated by a three-dimensional periodic solution. The projections of this periodic attractor onto three-dimensional subspaces are shown in Figure 32 and Figure 33.

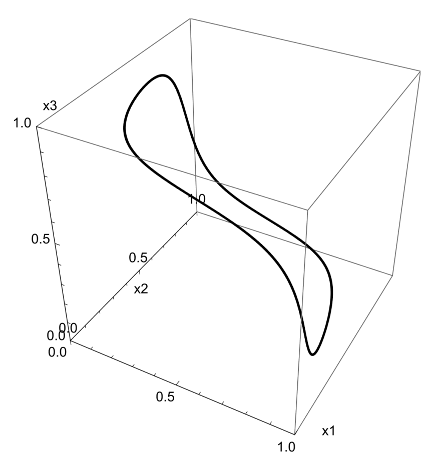

Figure 32: The projections of 6D trajectories to 3D subspace (x_1, x_2, x_3).

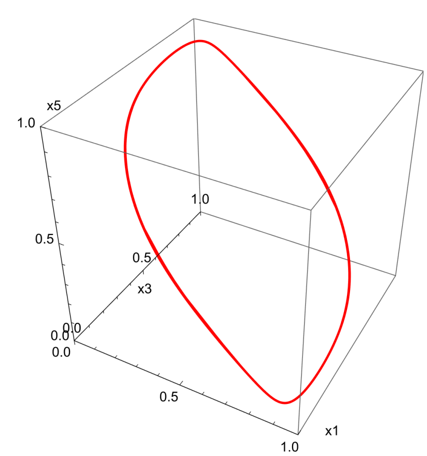

Figure 33: The projections of 6D trajectories to 3D subspace (x_1, x_3, x_5).

Consider the six-dimensional system (29) with the regulatory matrix (30), where $k_1 = 1$, $k_2 = 0.5$, $\mu_i = 5$, $\theta_i = \frac{k-1}{2}$.

The initial conditions are

$$x_1(0) = 0;\ x_2(0) = 0.4;\ x_3(0) = 0.1;\ x_4(0) = 0.2;\ x_5(0) = 0.1;\ x_6(0) = 0.1.$$

The projections of this periodic attractor onto three-dimensional subspaces are shown in Figure 34 and Figure 35.

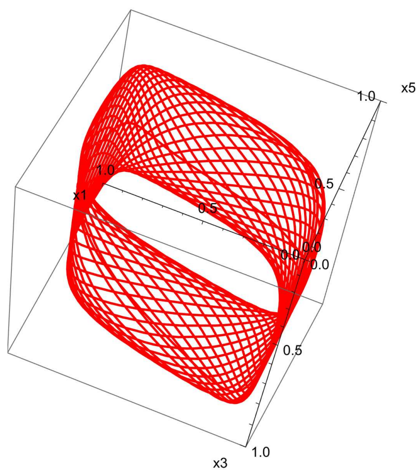

Figure 34: The projections of 6D trajectories to 3D subspace (x_1, x_3, x_5).

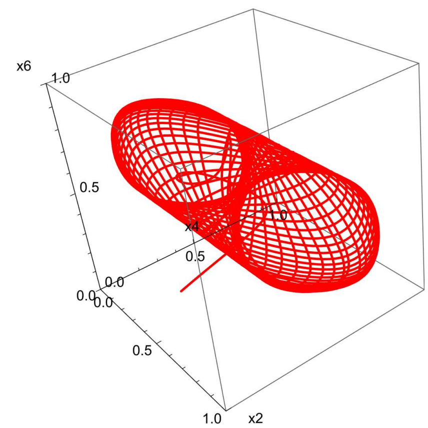

Figure 35: The projections of 6D trajectories to 3D subspace (x_2, x_4, x_6).

The respective six-dimensional system was studied in [68].

Example 2. We take the three-dimensional system (9) with the regulatory matrix (18), set of parameters (17) and initial conditions (19). It is depicted in Figure 7. The irregular behavior of three solutions can be seen in Figure 8.

Consider the six-dimensional system with the regulatory matrix

$$W = \begin{pmatrix} 0 & 1 & -5.64 & 0 & 0 & 0 \\ 1 & 0 & 0.1 & 0 & 0 & 0 \\ 1 & 0.02 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -5.64 \\ 0 & 0 & 0 & 1 & 0 & 0.1 \\ 0.5 & 0 & 0 & 1 & 0.02 & 0 \end{pmatrix} \tag{31}$$

and

$$\mu_1 = \mu_2 = \mu_4 = \mu_5 = 7, \mu_3 = \mu_6 = 13, v_1 = v_4 = 0.65, v_2 = v_5 = 0.42, v_3 = v_6 = 0.1,$$

$$\theta_1 = \theta_4 = 0.5, \theta_3 = \theta_5 = 0.3, \theta_3 = \theta_6 = 0.7.$$

The initial conditions are

$$x_1(1) = 0.68;\ x_2(1) = 0.45;\ x_3(1) = 0.15;\ x_4(1) = 0.68;\ x_5(1) = 0.45;\ x_6(1) = 0.15.$$

It would be uncoupled if the element w_{61} be zero. Then we would have a six-dimensional attractor which is the product of two identical three-dimensional attractors as in Figure

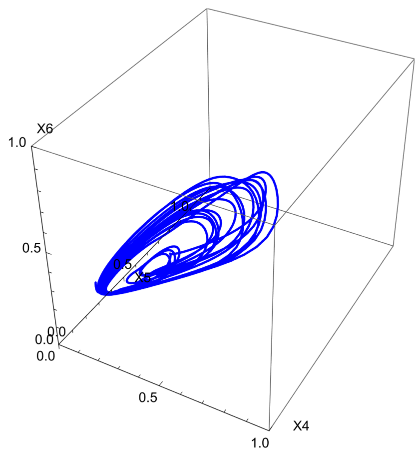

Figure 36: The projections of 6D trajectories to 3D subspace (x_4, x_5, x_6).

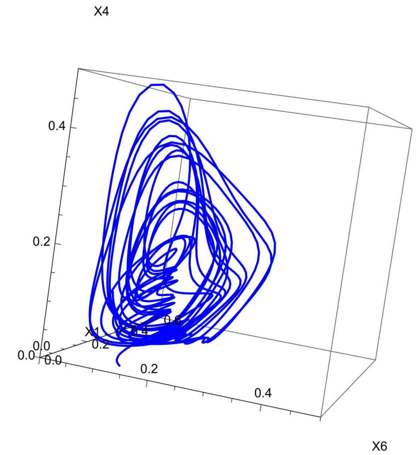

Figure 37: The projections of 6D trajectories to 3D subspace (x_1, x_4, x_6).

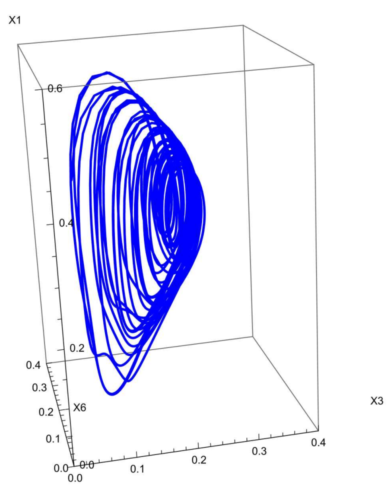

Figure 38: The projections of 6D trajectories to 3D subspace (x_1, x_3, x_6).

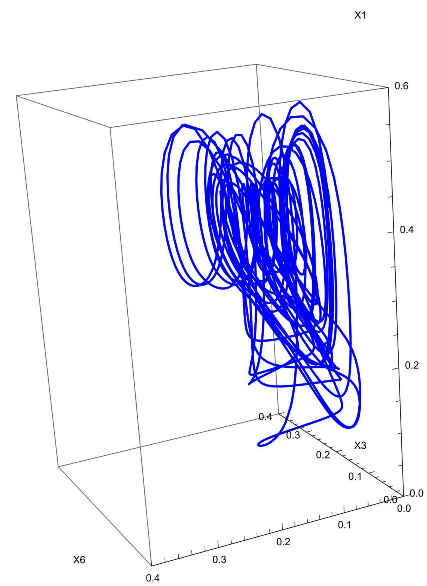

Figure 39: The projections of 6D trajectories to 3D subspace (x_1, x_3, x_6).

7. But w_{61} is set to 0.5. The six-dimensional system is coupled now. The new attractor exists and some of the three-dimensional projections are depicted in Figure 36 and Figure 37.

The solutions for system (29) with the matrix (31) are depicted in Figure 40 and Figure 41.

The graphs of solutions have irregular forms. They are different in Figure 40 and Figure 41 because of the non-zero element w_{61}.

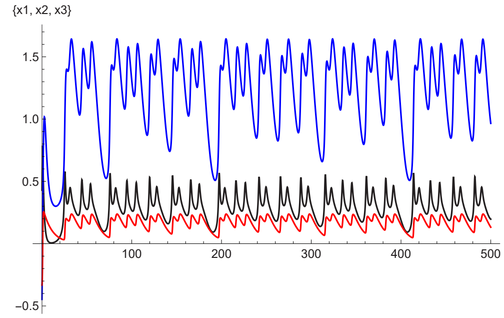

Figure 40: The graphs of solutions $x_i(t)$, $i = 1, 2, 3$, of the system (29) with the regulatory matrix (31), $w_{61} = 0.5$.

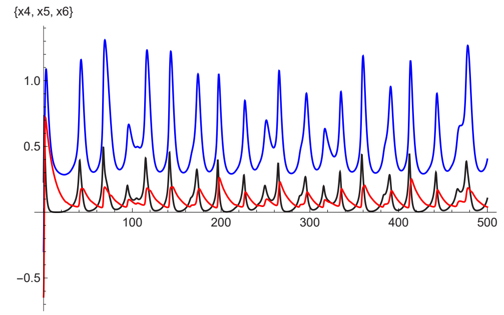

Figure 41: The graphs of solutions $x_i(t)$, $i = 4, 5, 6$, of the system (29) with the regulatory matrix (31), $w_{61} = 0.5$.

9 Sixty-dimensional (60D) systems

The network taken for the study is a realistic biological network, "T cells in large granular lymphocyte leukemia associated with blood cancer". A network model considered in [10],[89], contains 60 nodes and 195 regulatory edges. It was found in [89] that this network has three attractors, of which two correspond to two distinct cancerous states (denoted as C_1 and C_1) and one is associated with the normal state (denoted as N). The proper selection of the respective forty-eight parameters can drive the system to the normal state. The existence of needed parameter perturbation was acknowledged. The attractor network was considered and the main proposition was to arrange experimental adjustment of parameters in order to achieve the required goal.

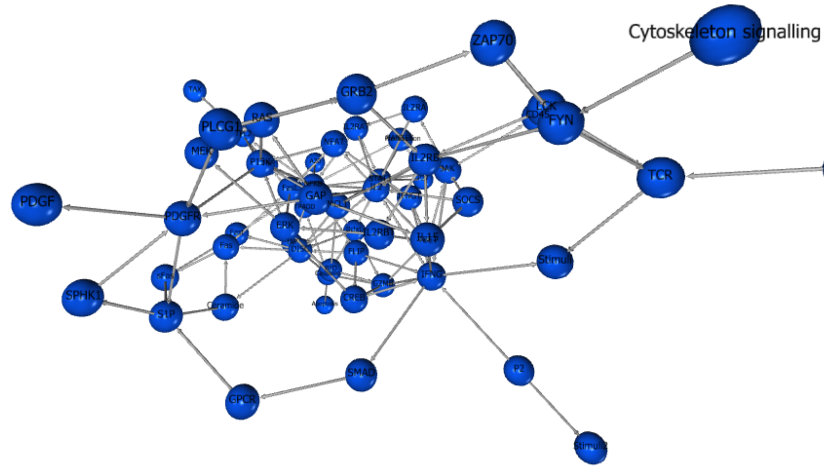

Figure 42: The graph of matrix 60×60.

To obtain this graph, the "Graphia" program was used. The matrix (43) was written in the program "Microsoft Excel".

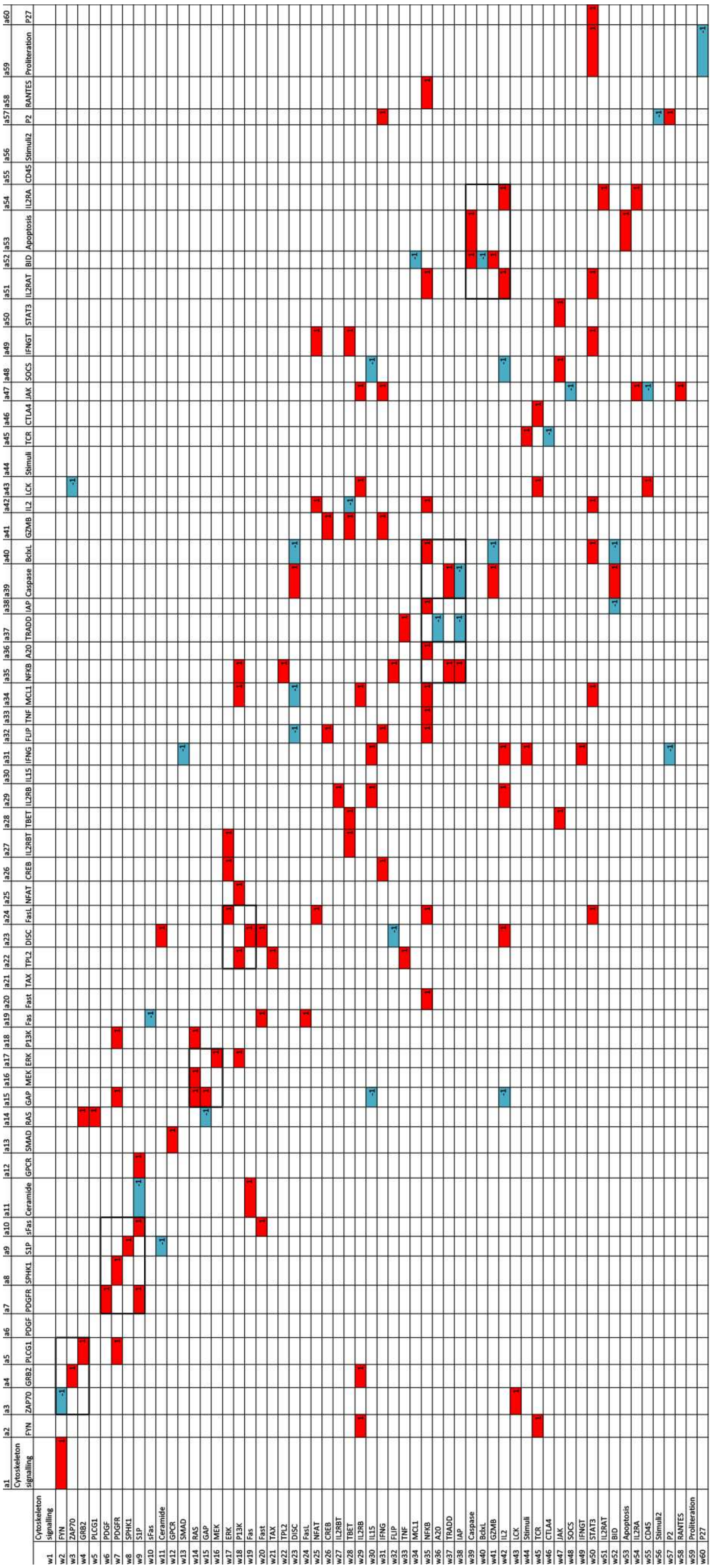

Figure 43: The regulatory matrix for Figure 42.

9.1 Subsystems

9.1.1 Three-dimensional systems

Example 1. Consider $\mu_1 = 5,\ \mu_2 = 15,\ \mu_3 = 5,\ v_1 = v_2 = v_3 = 1$ and $\theta_1 = 1.2,\ \theta_2 = 0.5,\ \theta_3 = -0.6$. The regulatory matrix of the system (9) is

$$W = \begin{pmatrix} w_2a_3 & w_2a_4 & w_2a_5 \\ w_3a_3 & w_3a_4 & w_3a_5 \\ w_4a_3 & w_4a_4 & w_4a_5 \end{pmatrix}, \tag{32}$$

where $w_2a_3 = -1, w_2a_4 = w_2a_5 = 0, w_3a_3 = w_3a_5 = 0, w_3a_4 = 1, w_4a_3 = w_4a_4 = 0, w_4a_5 = 1$. The nullclines are depicted in Figure 44. There are exactly three critical points.

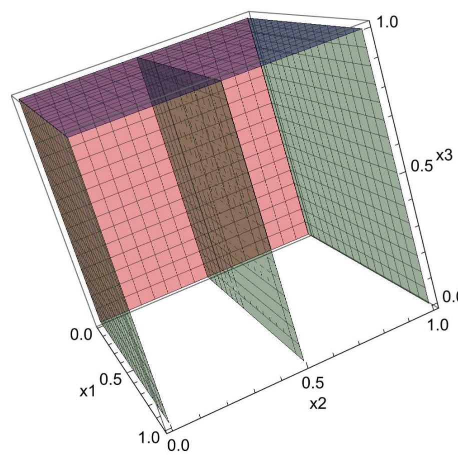

Figure 44: Visualization of nullclines (x_1 - red, x_2 - green, x_3 - blue) of the system(9) with the regulatory matrix (32).

The characteristic equation for critical point $(0.0024; 0.0006; 0.9997)$ is

$$-\lambda^3 + A\lambda^2 + B\lambda + C = 0, \tag{33}$$

where $A = -3.00215$, $B = -3.00419$ and $C = -1.00204$.

Solving the equation we have $\lambda_1 = -1.01218$, $\lambda_2 = -0.998321$ and $\lambda_3 = -0.991643$. The type of the critical point is a stable node.

The characteristic equation for critical point $(0.0024; 0.5; 0.9997)$ is (33), where $A = 0.739495$, $B = 4.5184$ and $C = 2.77883$.

Solving the equation we have $\lambda_1 = -1.01218$, $\lambda_2 = -0.998321$ and $\lambda_3 = 2.75$. The type of the critical point is a saddle.

The characteristic equation for critical point $(0.0024; 0.9994; 0.9997)$ is (33), where $A = -3.00215$, $B = -3.00419$ and $C = -1.00204$.

Solving the equation we have $\lambda_1 = -1.01218$, $\lambda_2 = -0.998321$ and $\lambda_3 = -0.991643$. The type of the critical point is a stable node.

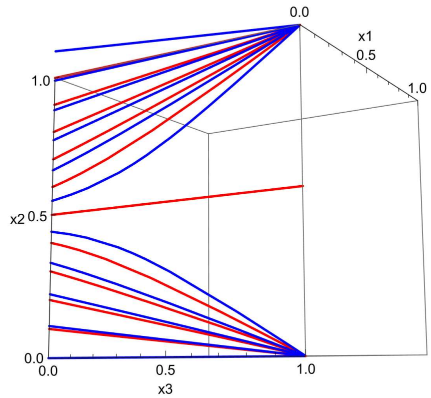

Figure 45: Visualization of two stable nodes and the saddle of the system(9) with the regulatory matrix (32).

9.1.2 Four-dimensional systems

Example 1. The regulatory matrix is

$$W = \begin{pmatrix} w_6a_7 & w_6a_8 & w_6a_9 & w_6a_{10} \\ w_7a_7 & w_7a_8 & w_7a_9 & w_7a_{10} \\ w_8a_7 & w_8a_8 & w_8a_9 & w_8a_{10} \\ w_9a_7 & w_9a_8 & w_9a_9 & w_9a_{10} \end{pmatrix},$$

where $w_6a_7 = w_7a_8 = w_8a_9 = w_9a_7 = w_9a_{10} = 1, w_6a_8 = w_6a_9 = w_6a_{10} = w_7a_7 = w_7a_9 = w_7a_{10} = w_8a_7 = w_8a_8 = w_8a_{10} = w_9a_8 = w_9a_9 = 0$ and $v_1 = v_2 = v_3 = v_4 = 1$, $\mu_1 = 5, \mu_2 = 15, \mu_3 = 5, \mu_4 = 5$, $\theta_1 = 1.2$, $\theta_2 = 0.5$, $\theta_3 = -0.6$, $\theta_4 = -0.2$.

The first critical point is $(0.0025; 0.00056; 0.9997; 0.9975)$. The standard linearization analysis provides the characteristic numbers $\lambda_1 = -0.998321$, $\lambda_2 = -0.991643$, $\lambda_3 = -0.987669$ and $\lambda_4 = -0.987513$. The type of the critical point is a 4D stable node.

The second critical point is $(0.00250369; 0.5; 0.999664; 0.997528)$. The standard linearization analysis provides the characteristic numbers $\lambda_1 = -0.998321$, $\lambda_2 = -0.987669$, $\lambda_3 = -0.987513$ and $\lambda_4 = 2.75$. The type of the critical point is a saddle.

The third critical point is $(0.00250369; 0.999443; 0.999664; 0.997528)$. The standard linearization analysis provides the characteristic numbers $\lambda_1 = -0.998321$, $\lambda_2 = -0.991643$, $\lambda_3 = -0.987669$ and $\lambda_4 = -0.987513$. The type of the critical point is a 4D stable node.

10 Conclusions

Main results of the Doctoral thesis are:

- Systems of orders two and three are considered with the regulatory matrices of different structures. The number and the character of critical points are considered.

- For three-dimensional systems and four-dimensional systems chaotic attractors were considered. Examples were constructed. In the thesis for Lyapunov exponents calculation the package “lce.m for Mathematica” was used. Another Wolfram Mathematica program “Lynch-DSAM.nb” was also used to check the correctness of Lyapunov exponents calculation.

- Formulas for characteristic numbers of critical points for four-dimensional systems were obtained. Examples of 4D systems with stable equilibria were constructed.
- Neuronal networks were considered and similarity with the corresponding ODE-type models was detected.
- Examples of 5D were constructed. These systems possess periodic attractors. The visualization of attractors of 5D by projecting them into lower dimension subspaces and considering graphs of components of solutions was made.
- Examples of 6D systems were constructed. These systems possess periodic attractors and exhibit irregular behavior of solutions. The visualization of attractors of 6D systems by projecting them into lower dimension subspaces and considering graphs of components of solutions was made.
- Sixty-dimensional system was considered. The graph of 60×60 matrix with the program Graphia was constructed. Some subsystems of the 60D system were considered.

The study of gene regulatory networks is important for human life and activity. Both for the treatment of various diseases such as leukemia, multiple sclerosis, and Alzheimer’s, and for describing problems and their solutions in economics, psychology, politics, and many other areas. The more are equations in a system, the more similar it is to the gene network that occurs in life. The main task is to continue the research and find methods for studying systems with a large number of equations.

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