DAUGAVPILS UNIVERSITY

Department of Physics and Mathematics

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A mathematical model for a class of networks in applications

Doctoral Thesis

Scientific supervisor
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The Doctoral Thesis and its summary are available at the library of Daugavpils University, Parades street 1 in Daugavpils and from http://du.lv/lv/zinatne/promocija/darbi

Comments are welcome. Send them to the secretary of the Promotion Council of mathematics, Parades street 1, Daugavpils, LV-5400, Tel. +371 26495316, e-mail: anita.sondore@du.lv

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Contents

1 General Information

Doctoral thesis contains 95 pages, 102 references, 123 figures, 6 tables.

Keywords and phrases: gene regulatory networks, mathematical modeling, phase portrait, periodic solutions, attractors, chaos.

Doctoral thesis

Object of research: a system of ordinary differential equations of the second and higher orders, used in models of gene regulatory networks.

Aims of research: to obtain results on properties of a special system of ordinary differential equations, making emphasis on attracting sets, their locations, dependence on built-in parameters and types of interrelation between elements. Special attention is paid to evolution of the system and prediction of its future behaviours.

The research tasks:

- overview of low-dimensional systems of ordinary differential equations (ODE), used in models of genetic regulatory networks (GRN);
- collecting information on equilibria (critical points) of attracting nature in low-dimensional systems;
- studying the nature of attractive equilibria in two-dimensional (2D) and three-dimensional (3D) systems;
- derivation of formulas for calculating the characteristic numbers of critical points in 2D and 3D systems;
- finding attractors, other than equilibria, in three-dimensional (3D) systems;
- considering examples of 3D systems, which have attractors in the form of stable periodic trajectories;
- considering examples of 3D systems, which exhibit chaotic behaviour of solutions;
- work with programs, detecting chaotic behavior on the basis of analysis of the Lyapunov exponents;
- considering systems of order four (4D), formulas for characteristic numbers of critical points;
- constructing examples of 4D systems, which have attractors in the form of stable equilibria;
- constructing examples of 4D systems, which have attractors in the form of stable periodic trajectories;

- considering examples of 4D systems, which exhibit chaotic behaviour of solutions;
- visualization of 4D attractors by projecting them on low-dimensional subspaces of the 4D phase space;
- considering examples of neuronal networks and detecting similarity in the corresponding ODE-type models;
- construction of examples of 5D and 6D systems which possess periodic attractors;
- considering examples of 6D systems, which exhibit irregular behaviour of solutions;
- visualization of attractors of 5D and 6D systems by projecting them into a lower dimension subspaces and considering graphs of components of solutions;
- overview of the results and outlining directions of future research.

Methods of research:

- classical techniques of mathematical analysis;
- comparison method;
- phase plane and phase space method;
- method of linearization around the trivial solution;
- perturbation method.

Main results: the results of the work were published in 23 scientific papers ([4], [48], [57]-[77]). Six of them ([4], [59], [67], [68], [69], [70]) have been published in the journals indexed in SCOPUS, three of them ([57], [66], [65]) were submitted to publish in the journals indexed in SCOPUS and two ([58], [71]) were submitted to publish in Web of Science journals. The results were communicated at several conferences of different levels:

1. Inna Samuilik, Nullcline method for research of GRN system critical points, The 78th Scientific Conference of the University of Latvia, (Riga, Latvia, February 28, 2020)
2. Felix Sadyrbaev, Svetlana Atslega, Inna Samuilik, On Controllability in Models of Biological Networks, VIII International Conference on Science and Technology, (Belgorod, Russia, September 24-25, 2020).
3. Inna Samuilik, Remark on four dimensional system arising in applications, The 79th Scientific Conference of the University of Latvia, (Riga, Latvia, February 26, 2021).
4. Felix Sadyrbaev, Inna Samuilik, Mathematical modelling of genetic regulatory networks, 2. International Baku Scientific Research Conference, (Baku, Azerbaijan, April 28-30, 2021).

5. Inna Samuilik, Felix Sadyrbaev, Mathematical modelling of evolution of multidimensional networks, 2. International Congress on Mathematics and Geometry, (Ankara, Turkey, May 20, 2021).

6. Svetlana Atslega, Felix Sadyrbaev, Inna Samuilik, On modelling of complex networks, 20th International Scientific Conference Engineering for Rural Development, (Jelgava, Latvia, May 27, 2021).

7. Inna Samuilik, Diana Ogorelova, Mathematical modelling of GRN using different sigmoidal functions, 1st International Symposium on Recent Advances in Fundamental and Applied Sciences, (Erzurum, Turkey, September 10-12, 2021).

8. Felix Sadyrbaev, Inna Samuilik, On the hierarchy of attractors in dynamical models of complex networks, 19th International Conference of Numerical Analysis and Applied Mathematics, (Rhodes, Greece, September 20-26, 2021)

9. Inna Samuilik, Felix Sadyrbaev, Valentin Sengileyev, Examples of periodic biological oscillators, International Conference "Differential Equations, Mathematical Modeling and Computational Algorithm", (Belgorod, Russia, October 25-29, 2021).

10. Felix Sadyrbaev, Inna Samuilik, Albert Silvans, On mathematical models of evolving networks, International Conference "Differential Equations and Related Topics, 24th joint session of Moscow Mathematical Society and I.G.Petrovskii Seminar", (Moscow, Russia, December 26-30, 2021).

11. Inna Samuilik, Felix Sadyrbaev, Diana Ogorelova, Mathematical modeling of three-dimensional genetic regulatory networks using different sigmoidal functions, International liberty interdisciplinary studies conference, (NewYork, ASV, January 16-17, 2022).

12. Inna Samuilik, On a four-dimensional system of differential equations related to the theory of gene regulatory networks, The 80th Scientific Conference of the University of Latvia, (Riga, Latvia, February 25, 2022).

13. Inna Samuilik, Felix Sadyrbaev, A Note on Attractor Selection, The 5th International Conference on Networking, Intelligent Systems and Security, (Bandung, Indonesia, March 30-31, 2022).

14. Inna Samuilik, Felix Sadyrbaev, Svetlana Atslega, Mathematical modelling of nonlinear dynamic systems, 21st International Scientific Conference Engineering for Rural Development, (Jelgava, Latvia, May 25-27, 2022).

15. Inna Samuilik, Genetic engineering-construction of a network of four dimensions with a chaotic attractor, 58th International JVE Conference,(Ventspils, Latvia, August 25-26, 2022).

2 Preface

The Theory of ordinary differential equations (ODE in short) has emerged from applications and serves applications. First, problems of mechanical motion have led to the second-order ODE. Recall Isaac Newton's second law: force is equal to the product of mass and acceleration. After that investigations by several renowned scientists yielded the linear theory, stability theory, theory of periodic motions, etc. Closer to our times, new branches of the theory have appeared. Among them, the theory of boundary value problems (BVP) for ODE took a significant place. Riga and its universities have strong traditions in this field. Pierce Bohl is known for the creation of fixed point theorems for integral and differential equations. In the middle of the 20th century, Anatoliy Myshkis had arrived in Riga to teach students, among them were Yurii Klokov and Arnol'd Lepin. Y. Klokov went to Moscow State University, where he finished his doctoral studies under the supervision of Professor Vladimir Nemyckii. His dissertation was devoted to BVP at infinite intervals. Y. Klokov started to work in Riga Civil Aviation Engineers Institute and was recognized by students as the best teacher. After some time he was invited to join the new-born Computational center of Latvian State University, where he worked for a long time. A. Lepin was conducting his research under the supervision of Professor Anatoliy Myshkis. His dissertation also was devoted to BVP for nonlinear differential equations. Y. Klokov and A. Lepin headed the scientific division which studied BVP and related problems. This research is continued now in the Institute of Mathematics and Computer Science of the University of Latvia. Y. Klokov and A. Lepin had many doctorate students and descendants. Some of them are still actively working in the field of differential equations, applications, and mathematical modeling. Another direction in the theory of differential equations was established by Professor Linard Reizins. This direction was aimed at the qualitative studies of ordinary differential equations. The problems of structural stability and classification of critical points for higher order equations were at the center of his and his student's studies. It appears that Riga and Latvia had and still have long-standing traditions in the field of the theory of ODE.

In 2015, when joining the group of Professor Alexander Shostak for the studies in the field of telecommunication networks in the framework of a Europen project, the group of mathematicians from Riga and Daugavpils was attracted by a new kind of problems, where ordinary differential equations were involved. In the theory of telecommunication networks very active group of Japanese mathematicians, among them, Yuki Koizumi, Masayuki Murata, and others, proposed to use in the design of telecommunication networks the principles of self-organization, that could be found in Nature. It was pointed out, that in living organisms in any cell there exists a genetic regulatory network (GRN in short), responsible, among others, for reactions to changes in the environment. It was emphasized, that the mathematical model for GRN, which uses a system of ODE, can be used also for the management and control of telecommunication networks. The so-called Virtual Network Topology was proposed for the organization of a set of lightpaths, to establish a mechanism for quick response and rearrangement of a telecommunication network in bad conditions. A system of ODE, governing this process of rearrangement, had attracted the attention of researchers in Institute of Mathematics and Computer Science, University of Latvia and Daugavpils University. It appears that accumulated previous knowledge in the theory of ODE and experience in the studies of ODE can be applied to new kind of problems. It was the starting point of research in this direction.

The system in the center of these studies is not easy, but it is in some sense symmetrical. It contains n ordinary differential equations of the form $X' = F(WX - \theta) - VX$, where the vector X is for unknowns, F is a vector of the so-called sigmoidal functions, W is $n \times n$ matrix (it is called regulatory one), θ and V are the parameters. This system will be denoted by S in this preface. It appeared first in the paper by Cowan-Vilson in the study of neuronal networks of the human brain. It was used to model genetic networks, and the meaning of X was different. It was associated with the protein expression of any gene. By protein expression genes communicate with each other. Affecting a single gene can affect the whole network. In system S the linear part describes the natural decay of a network, where there is no interrelation between genes. Some authors introduce in this model also other factors, such as stress. Generally speaking, the object of investigation is a multi-parameter autonomous quasi-linear system of ordinary differential equations. To the best of the author's knowledge, this system was not studied sufficiently for dimensions three and higher. One of the possible reasons is the lacking of theoretical results for systems of this kind. In the last decade, many papers had appeared, interpreting system S specifically. In the remarkable papers [10], [89] by Cornelius et al and Le-Zhi Wang et al. the X vector was interpreted as the state vector for the genetic network. This vector is time-dependent, $X(t)$, and it goes to its limiting set or point. It is to be mentioned, that the phase space for the system S has a time-invariant set Q with the property: any trajectory of the system S which enters Q never escapes it. The existence of attracting sets in Q follows. In the above interpretation, some diseases can be treated having in mind that the respective state-vector $X(t)$, therefore, is forced to go to the "wrong" attractor. Since system S contains a lot of parameters, some of them are adjustable and can be used to manage and control a network. Treatment of a disease then means redirecting of "bad" trajectory to a "normal" attractor.

The above considerations were good motivations for the study of system S and its attractor. The proposed promotional work contains achievements in this direction. Let us list them.

1. The 2-dimensional systems were studied, using the nullclines method;

2. The 3-dimensional systems were studied, using the nullcline method and extensive computational research; the main results are a) formulas for critical points of a 3D system; b) multiple examples of periodic attractors;

3. The 4-dimensional systems were studied, using previously obtained results for 2-dimensional systems; uncoupled 4D systems were constructed of two independent 2D-systems and various resulting combinations of attractors were studied; the main results are a) formulas for critical points; b) periodic attractors for uncoupled 4D systems; c) examples of periodic attractors; d) examples of perturbed 4D systems, which are no longer uncoupled; some conclusions were made about attractors in perturbed systems; d) an irregular behavior of solutions, tending to a 4D attractor, was observed;

4. Some examples of the 5-dimensional systems were examined;

5. The 6-dimensional systems were studied; the main results are a) examples of 6D-systems which were constructed of previously investigated three 2D systems; the resulting system can have attractors of periodic nature; b) examples of 6D-systems

which were constructed of previously investigated two 3D systems; the resulting system can have attractors of periodic nature; c) examples of perturbed, and therefore coupled, 6D systems were examined; some observations on the behavior of solutions were made;

6. The 60-dimensional system was considered.

Generally, the work contains mainly computationally obtained results concerning systems of the form S, their phase space, examples of attractors and many related facts. The collection obtained results lay the foundation for further research into gene network models to understand them and develop methods of management and control.

3 Gene regulatory network

Gene regulatory networks (GRN in short) exist in any cell of any living organism. GRN regulates reactions to changes in the environment, controls the development of a cell, and manages the functioning of any kind. Elements of GRN, called genes, can influence other genes by sending proteins [43]. As a result of such influence, other genes can be activated or inhibited.
Attempts to mathematically model the functioning of GRN are multiple, using various mathematical objects and tools [26],[87]. To describe the evolution of a network, the most appropriate approach is using differential equations.

The typical system is of the form

$$X' = F(WX - \theta) - vX,$$

where X is the network state vector, F is a sigmoid nonlinearity with argument, transformed by multiplication with the regulatory matrix W, vX is a natural decay in absence of F.

Definition 3.1. *A sigmoid function is a mathematical continuous function with the domain over all* $\mathbb{R}$ *having a characteristic "S-shaped" curve or sigmoid curve. The range of a sigmoid function is* $(0, 1)$*.*

The name "Sigmoid" comes from the Greek letter Sigma, and when graphed, appears as a sloping "S" across the y-axis. In general, a sigmoid function is monotonic and has a first derivative.

Sigmoid function's examples.

- The logistic function or logistic curve $f(z) = \dfrac{1}{1 + e^{-\mu(z-\theta)}}$. The sigmoid logistic function was introduced in a series of three papers by Pierre Francois Verhulst between 1838 and 1847, who devised it as a model of population growth by adjusting the exponential growth model.

- Gompertz curve or Gompertz function $f(z) = e^{-e^{-\mu(z-\theta)}}$. The Gompertz model is well known and widely used in many aspects of biology. It has been frequently used to describe the growth of animals and plants, as well as the number or volume of bacteria and cancer cells.

- The inverse trigonometric functions $f(z) = \dfrac{2}{\pi}\arctan[\mu(z+\theta)]$. The inverse trigonometric functions are widely used in engineering, navigation, physics, and geometry.

- The hyperbolic tangent function $f(z) = \dfrac{1}{2}\tanh[z-1] + \dfrac{1}{2}$. The hyperbolic tangent function is used in Artificial Neural networks.

- Hill function $f(z) = \dfrac{z^\mu}{\theta^\mu + z^\mu}$ was introduced by Archibald Hill in 1910 to describe the binding of oxygen to hemoglobin. Hill function is widely used in pharmacology, biochemistry and physiology.

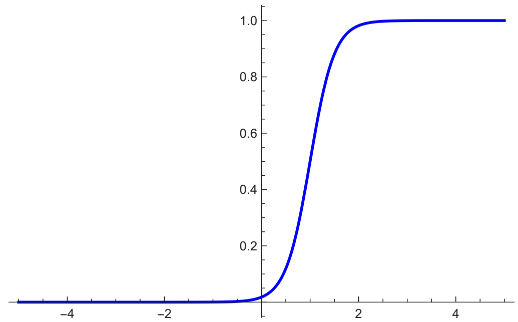

Figure 1: The sigmoid logistic function.

Definition 3.2. *A dynamical system is a system of equations describing the time evolution of one or more dependent variables. Equations of motion can be modelled as differential equations and difference equations [34].*

Consider the n-dimensional dynamical system

$$\begin{cases} \dfrac{dx_1}{dt} = \dfrac{1}{1+e^{-\mu_1(w_{11}x_1+w_{12}x_2+\ldots+w_{1n}x_n-\theta_1)}} - v_1x_1, \\ \dfrac{dx_2}{dt} = \dfrac{1}{1+e^{-\mu_2(w_{21}x_1+w_{22}x_2+\ldots+w_{2n}x_n-\theta_2)}} - v_2x_2, \\ \ldots \\ \dfrac{dx_n}{dt} = \dfrac{1}{1+e^{-\mu_n(w_{n1}x_1+w_{n2}x_2+\ldots+w_{nn}x_n-\theta_n)}} - v_nx_n, \end{cases} \tag{1}$$

where $\mu_n > 0$, θ_n and $v_n > 0$ are parameters and the coefficients w_{ij} are entries of the so-called regulatory matrix

$$W = \begin{pmatrix} w_{11} & w_{12} & \ldots & w_{1n} \\ w_{21} & w_{22} & \ldots & w_{2n} \\ \ldots & & & \\ w_{n1} & w_{n2} & \ldots & w_{nn} \end{pmatrix}. \tag{2}$$

The parameters of the GRN have the following biological interpretations:

- v_i - degradation of the i-th gene expression product;
- w_{ij} - the connection weight or strength of control of gene j on gene i. Positive values of w_{ij} indicate activating influences while negative values define repressing influences;
- θ_i - influence of external input on gene i, which modulates the gene's sensitivity of response to activating or repressing influences.

Definition 3.3. *The j'th nullcline is the geometric shape for which* $\dfrac{dx_j}{dt} = 0$. *The critical points of the system are located where all of the nullclines intersect. In a two-dimensional linear system, the nullclines can be represented by two lines on a two-dimensional plot; in a general two-dimensional system they are arbitrary curves.*

The nullclines for the system (1) are

$$\begin{cases} x_1 = \dfrac{1}{v_1}\dfrac{1}{1+e^{-\mu_1(w_{11}x_1+w_{12}x_2+...+w_{1n}x_n-\theta_1)}}, \\ \\ x_2 = \dfrac{1}{v_2}\dfrac{1}{1+e^{-\mu_2(w_{21}x_1+w_{22}x_2+...+w_{2n}x_n-\theta_2)}}, \\ \\ ... \\ x_n = \dfrac{1}{v_n}\dfrac{1}{1+e^{-\mu_n(w_{n1}x_1+w_{n2}x_2+...+w_{nn}x_n-\theta_n)}}. \end{cases}$$

The linearized system for analyzing critical points is

$$\begin{cases} u_1' = -v_1u_1 + \mu_1w_{11}g_1u_1 + \mu_1w_{12}g_1u_2 + ... + \mu_1w_{1n}g_1u_n, \\ u_2' = -v_2u_2 + \mu_2w_{21}g_2u_1 + \mu_2w_{22}g_2u_2 + ... + \mu_2w_{2n}g_2u_n, \\ ... \\ u_n' = -v_nu_n + \mu_nw_{n1}g_nu_1 + \mu_nw_{n2}g_nu_2 + ... + \mu_nw_{nn}g_nu_n, \end{cases}$$

where

$$g_1 = \frac{e^{-\mu_1(w_{11}x_1^*+w_{12}x_2^*+...+w_{1n}x_n^*-\theta_1)}}{[1+e^{-\mu_1(w_{11}x_1^*+w_{12}x_2^*+...+w_{1n}x_n^*-\theta_1)}]^2},$$

$$g_2 = \frac{e^{-\mu_2(w_{21}x_1^*+w_{22}x_2^*+...+w_{2n}x_n^*-\theta_2)}}{[1+e^{-\mu_2(w_{21}x_1^*+w_{22}x_2^*+...+w_{2n}x_n^*-\theta_2)}]^2},$$

$$...$$

$$g_n = \frac{e^{-\mu_n(w_{n1}x_1^*+w_{n2}x_2^*+...+w_{nn}x_n^*-\theta_n)}}{[1+e^{-\mu_n(w_{n1}x_1^*+w_{n2}x_2^*+...+w_{nn}x_n^*-\theta_n)}]^2}$$

and $(x_1^*, x_2^*, ..., x_n^*)$ is a critical point.

One has

$$A - \lambda I = \begin{vmatrix} \mu_1w_{11}g_1 - v_1 - \lambda & \mu_1w_{12}g_1 & ... & \mu_1w_{1n}g_1 \\ \mu_2w_{21}g_2 & \mu_2w_{22}g_2 - v_2 - \lambda & ... & \mu_2w_{2n}g_2 \\ & & & \\ ... & & & \\ & & & \\ \mu_nw_{n1}g_n & \mu_nw_{n2}g_n & ... & \mu_nw_{n3}g_n - v_n - \lambda \end{vmatrix}$$

The characteristic values of a given critical point can be found by solving the equation $A - \lambda I$ with respect to λ.

3.1 The parameter μ

Depending on the value of the parameter μ, the form of the sigmoid function changes. For very small μ values the sigmoid function has a characteristic "S-shaped" curve or sigmoid curve (see Figure 2). The parameter μ increases and the graph of the sigmoid function changes from "S-shaped" curve to a piecewise linear curve (see Figure 3). Further, the parameter μ continues to increase, but the graph of the sigmoid function is also the piecewise linear curve (see Figure 4). Large values of μ make nullcline the piecewise linear.

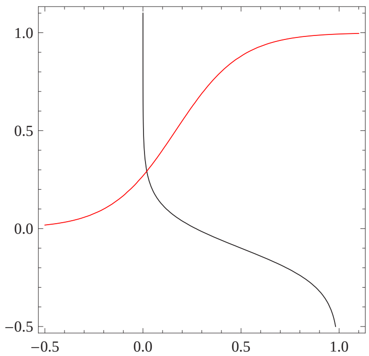

Figure 2: $\mu = 2, \theta = 0.5$; $w_{11} = 0, w_{12} = -5$, $w_{21} = 3, w_{22} = 0$

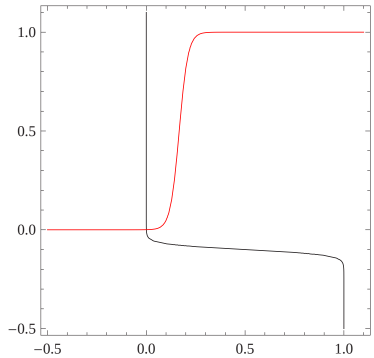

Figure 3: $\mu = 15, \theta = 0.5$; $w_{11} = 0, w_{12} = -5$, $w_{21} = 3, w_{22} = 0$

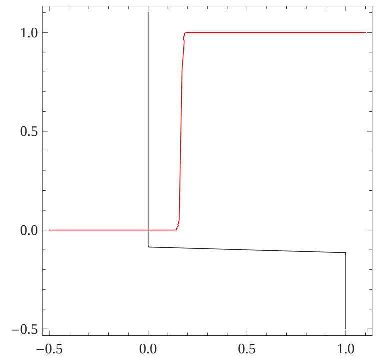

Figure 4: $\mu = 100, \theta = 0.5$; $w_{11} = 0, w_{12} = -5$, $w_{21} = 3, w_{22} = 0$

4 Two-dimensional (2D) systems

Consider the system

$$\begin{cases} \dfrac{dx_1}{dt} = \dfrac{1}{1 + e^{-\mu_1(w_{11}x_1 + w_{12}x_2 - \theta_1)}} - v_1 x_1, \\[2ex] \dfrac{dx_2}{dt} = \dfrac{1}{1 + e^{-\mu_2(w_{21}x_1 + w_{22}x_2 - \theta_2)}} - v_2 x_2, \end{cases} \tag{3}$$

where μ_i and v_i are positive.

System (3) contains ten parameters $w_{ij}, \mu_i, \theta_i, v_i$. Changing any of these parameters can essentially affect the properties of the system and solutions. The construction of the characteristic equation is a nontrivial task.

The argument z of the sigmoid function is transformed by the regulatory (coefficient) matrix

$$W = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}. \tag{4}$$

This matrix describes the interrelation of elements x_i of a network. The structure of W affects the properties of the system and its solutions.

The nullclines are given by the equations

$$\begin{cases} x_1 = \dfrac{1}{v_1} \dfrac{1}{1 + e^{-\mu_1\,(w_{11}x_1 + w_{12}x_2 - \theta_1)}}, \\[2ex] x_2 = \dfrac{1}{v_2} \dfrac{1}{1 + e^{-\mu_2\,(w_{21}x_1 - w_{22}x_2 - \theta_2)}}. \end{cases} \tag{5}$$

The function $f(z) = \dfrac{1}{1 + e^{-\mu z}}$ is a sigmoid and it has the range of values $(0, 1)$. Therefore the first nullcline is in the strip $\{(x_1, x_2) : 0 < x_1 < \frac{1}{v_1}, x_2 \in \mathbb{R}\}$ and the second one is in the strip $\{(x_1, x_2) : x_1 \in \mathbb{R}, 0 < x_2 < \frac{1}{v_2}\}$. Therefore all critical points are located in the rectangle $Q := \{(x_1, x_2) : 0 < x_1 < \frac{1}{v_1}, 0 < x_2 < \frac{1}{v_2}\}$.

Proposition 4.1. *There exists at least one critical point for the system* (3).

Proposition 4.2. *The vector field* (x_1', x_2') *defined by (3) is directed inward on the border of the rectangle* Q.

Proof. By inspection of the right-hand sides in (3), taking into account (5) and the range of values of the functions $f_1(z)$ and $f_2(z)$. □

Corollary 4.1. *Any trajectory of the system (3), entering* Q *will stay there for* $t > 0$.

4.1 Linearized system

For the analysis of critical points, we need the linearized system. It takes the form

$$\begin{cases} u_1' = -v_1 u_1 + \mu_1 w_{11} g_1 u_1 + \mu_1 w_{12} g_1 u_2, \\ u_2' = -v_2 u_2 + \mu_2 w_{21} g_2 u_1 + \mu_2 w_{22} g_2 u_2, \end{cases}$$

where

$$g_1 = \frac{e^{-\mu_1(w_{11}x_1^* + w_{12}x_2^* - \theta_1)}}{[1 + e^{-\mu_1(w_{11}x_1^* + w_{12}x_2^* - \theta_1)}]^2},$$

$$g_2 = \frac{e^{-\mu_2(w_{21}x_1^* + w_{22}x_2^* - \theta_2)}}{[1 + e^{-\mu_2(w_{21}x_1^* + w_{22}x_2^* - \theta_2)}]^2}$$

and (x_1^*, x_2^*) is a critical point under consideration. Notice that $0 < g_i < 0.25$ for $i = 1, 2$.

$$A = \begin{vmatrix} \mu_1 w_{11} g_1 - v_1 & \mu_1 w_{12} g_1 \\ \mu_2 w_{21} g_2 & \mu_2 w_{22} g_2 - v_2 \end{vmatrix}$$

$$A - \lambda I = \begin{vmatrix} \mu_1 w_{11} g_1 - v_1 - \lambda & \mu_1 w_{12} g_1 \\ \mu_2 w_{21} g_2 & \mu_2 w_{22} g_2 - v_2 - \lambda \end{vmatrix}$$

and the characteristic equation is

$$\begin{aligned} &\det|A - \lambda I| = (\mu_1 w_{11} g_1 - v_1 - \lambda)(\mu_2 w_{22} g_2 - v_2 - \lambda) - (\mu_2 w_{21} g_2)(\mu_1 w_{12} g_1) = \\ &\mu_1\mu_2 w_{11} w_{22} g_1 g_2 - \mu_1 w_{11} g_1 v_2 - \mu_1 w_{11} g_1 \lambda - \mu_2 w_{22} g_2 v_1 + v_1 v_2 + v_1 \lambda - \mu_2 w_{22} g_2 \lambda + \\ &v_2 \lambda + \lambda^2 - \mu_1\mu_2 w_{12} w_{21} g_1 g_2 = \lambda^2 + (v_1 + v_2 - \mu_1 w_{11} g_1 - \mu_2 w_{22} g_2)\lambda + \\ &\mu_1\mu_2 w_{11} w_{22} g_1 g_2 - \mu_1 w_{11} g_1 v_2 - \mu_2 w_{22} g_2 v_1 - \mu_1\mu_2 w_{12} w_{21} g_1 g_2 + v_1 v_2 = 0. \end{aligned}$$

To simplify we can write the characteristic equation as

$$\lambda^2 + B\lambda + C = 0,$$

where

$$B = v_1 + v_2 - \mu_1 w_{11} g_1 - \mu_2 w_{22} g_2,$$

$$C = \mu_1\mu_2 w_{11} w_{22} g_1 g_2 - \mu_1 w_{11} g_1 v_2 - \mu_2 w_{22} g_2 v_1 - \mu_1\mu_2 w_{12} w_{21} g_1 g_2 + v_1 v_2.$$

4.2 Critical points

An attractor is a set of points in phase space that represent a stable set of final dynamics for the system [81]. These dynamics are final in three senses. First, once the state of the system or model is in this set, it does not leave this set. Second, all points of the set are reached. Finally, any trajectory starting near enough to the attractor approaches the attractor. For a simple continuous-time model, depending on the parameters, the attractor can be a single point (a critical point), two points (a two-point cycle), four, eight, or a larger finite number of points (a more complex cycle), a closed curve, or a chaotic attractor [23].

Definition 4.1. *An attractor is the limiting trajectory of the representing point in the phase space, to which all initial modes tend [3].*

Each attractor has a basin of attraction that contains all the initial conditions which will generate trajectories joining asymptotically this attractor [2].

Definition 4.2. *A self-excited attractor has a basin of attraction that is associated with an unstable critical point [36].*

Self-excited attractors can be localized numerically by the standard computational procedure: after a transient process, a trajectory that starts in the neighborhood of an unstable critical point (from a point on its unstable manifold) is attracted to the attractor and traces it [36]. Classical attractors in Van der Pol, Rossler, Chua dynamical systems are self-excited.

Definition 4.3. *An attractor is called a self-excited attractor if its basin of attraction intersects an open neighborhood of a critical point, otherwise it is called a hidden attractor [11].*

For a hidden attractor, its attraction basin is not connected with an unstable critical point. For example, hidden attractors can be found in systems without equilibria or with stable equilibria [11].

Definition 4.4. *A limit cycle is a closed trajectory in phase space having the property that at least one other trajectory spirals into it, either as time approaches to infinity or as time approaches to negative infinity [17].*

Proposition 4.3. *A limit cycle can exist in nonlinear systems of ODE, the number of equations in which is $n \geq 2$.*

Proposition 4.4. *A closed trajectory has a critical point in its interior in space $\mathbb{R}^2$.*

If it is a stable state of equilibrium (critical point), the attractor of the system will be just a fixed point. If it is a stable periodic motion, then the attractor will be a closed curve, called the limit cycle [3].

Set $w_{11} = w_{22} = 0$. The regulatory matrix is

$$W = \begin{pmatrix} 0 & w_{12} \\ w_{21} & 0 \end{pmatrix}$$

and the system of differential equations takes the form

$$\begin{cases} x_1' = \dfrac{1}{1+e^{-\mu_1(w_{12}x_2-\theta_1)}} - v_1x_1, \\ x_2' = \dfrac{1}{1+e^{-\mu_2(w_{21}x_1-\theta_2)}} - v_2x_2. \end{cases}$$

The characteristic equation is

$$\lambda^2 + B\lambda + C = 0, \tag{6}$$

where

$$B = v_1 + v_2,$$
$$C = -\mu_1\mu_2 w_{12}w_{21}g_1g_2 + v_1v_2.$$

4.2.1 Case 1

$$B^2 > 4C,\ \lambda_{1,2} = -\frac{B}{2} \pm \sqrt{\frac{B^2}{4} - C} > 0$$
$$B < 0 \Rightarrow \lambda_1 > 0 \Rightarrow \lambda_2 > 0.$$

Proposition 4.5. *If $B^2 > 4C$, $\lambda_{1,2} > 0$. The case stable node is impossible.*

Consider

$$w_{11} = w_{22} = 0,\ B = v_1 + v_2,\ C = v_1v_2 - \mu_1\mu_2w_{12}w_{21}g_1g_2.$$

4.2.2 Case 2

$$B^2 > 4C,\ B > 0,\ \lambda_{1,2} = -\frac{B}{2} \pm \sqrt{\frac{B^2}{4} - C}$$

Case 2.1.

$$C > 0,\ 0 < \frac{B^2}{4} - C < \frac{B^2}{4} \Rightarrow \lambda_2 = -\frac{B}{2} + \sqrt{\frac{B^2}{4} - C} < 0,$$

$$\Rightarrow \lambda_1 = -\frac{B}{2} - \sqrt{\frac{B^2}{4} - C} < 0.$$

Proposition 4.6. *If $B^2 > 4C$, $C > 0 \Rightarrow \lambda_{1,2} < 0$, then only stable node is possible.*

Consider

$$w_{11} = w_{22} = 0,\ B^2 > 4C,\ C > 0$$
$$(v_1 + v_2)^2 > 4(v_1v_2 - \mu_1\mu_2w_{12}w_{21}g_1g_2), \quad v_1v_2 - \mu_1\mu_2w_{12}w_{21}g_1g_2 > 0,$$
$$v_1 = v_2 = 1, \quad 4 > 4(1 - \mu_1\mu_2w_{12}w_{21}g_1g_2).$$

Example 1. Consider $\mu_1 = 5, \mu_2 = 10,\ v_1 = v_2 = 1$ and $\theta_1 = 0.2, \theta_2 = 0.25$. The regulatory matrix is

$$W = \begin{pmatrix} 0 & -4 \\ -2 & 0 \end{pmatrix}. \tag{7}$$

The characteristic equation for the critical point $(0.2674; 0.0004)$ is (6), where $B = 2$, $C = 0.969076$.

Solving the equation we have $\lambda_1 = -1.17585$ and $\lambda_2 = -0.824148$. The type of the critical point is a stable node.

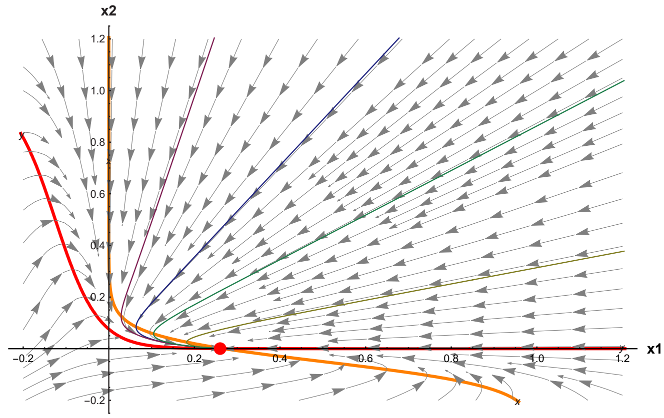

Figure 5: The phase portrait for the system (3) with the regulatory matrix (7).

Case 2.2.

$$C < 0,\ \lambda_1 < 0,\ \lambda_2 = -\frac{B}{2} + \sqrt{\frac{B^2}{4} - C} > 0.$$

Proposition 4.7. *If $B^2 > 4C, C < 0 \Rightarrow \lambda_1 < 0, \lambda_2 > 0$, then only saddle is possible.*

Consider

$$\omega_{11} = \omega_{22} = 0,\ v_1 = v_2 = 1,\ \mu_1 = \mu_2 = \mu,$$

$$4 > 4(1 - \mu^2 w_{12} w_{21} g_1 g_2), \quad 0 > -\mu^2 w_{12} w_{21} g_1 g_2.$$

Example 2. Consider $\mu_1 = \mu_2 = 5$, $v_1 = v_2 = 1$ and $\theta_1 = 1.2, \theta_2 = 0.8$. The regulatory matrix is

$$W = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}. \tag{8}$$

The characteristic equation for the critical point $(0.66; 0.33)$ is (6), where $B = 2$, $C = -3.98$.

Solving the equation we have $\lambda_1 = -3.23$ and $\lambda_2 = 1.23$. The type of the critical point is a saddle.

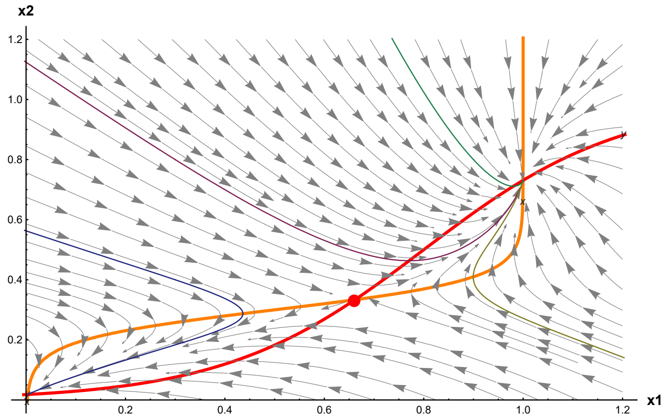

Figure 6: The phase portrait for the system (3) with the regulatory matrix (8).

4.2.3 Case 3

$$B^2 < 4C,\ \lambda_{1,2} = -\frac{B}{2} \pm \sqrt{\frac{B^2}{4} - C} \in \mathbb{C}.$$

Consider

$$\omega_{11} = \omega_{22} = 0,\ (v_1 + v_2)^2 < 4(v_1 v_2 - \mu_1 \mu_2 w_{12} w_{21} g_1 g_2),$$
$$v_1^2 + 2v_1 v_2 + v_2^2 < 4v_1 v_2 - 4\mu_1 \mu_2 w_{12} w_{21} g_1 g_2,$$
$$(v_1 - v_2)^2 < -4\mu_1 \mu_2 w_{12} w_{21} g_1 g_2.$$

Proposition 4.8. *If $(v_1 - v_2)^2 < -4\mu_1 \mu_2 w_{12} w_{21} g_1 g_2$, then all critical points are focuses.*

Corollary 4.2. *The necessary condition for all critical points to be focuses is $w_{12} w_{21} < 0$.*

4.2.4 Case 4

$$B^2 < 4C,\ -\frac{B}{2} > 0 \Rightarrow B < 0.$$

Proposition 4.9. *If $w_{11} = w_{22} = 0$, then $B > 0$ and no critical point is an unstable focus.*

$$B^2 - 4C = (v_1 + v_2 - \mu_1 w_{11} g_1 - \mu_2 w_{22} g_2)^2 -$$
$$-4(\mu_1 \mu_2 w_{11} w_{22} g_1 g_2 - \mu_1 w_{11} g_1 v_2 - \mu_2 w_{22} g_2 v_1 - \mu_1 \mu_2 w_{12} w_{21} g_1 g_2 + v_1 v_2) =$$
$$= v_1^2 + 2v_1 v_2 - 2v_1 \mu_1 w_{11} g_1 - 2v_1 \mu_2 w_{22} g_2 + v_2^2 - 2v_2 \mu_1 w_{11} g_1 - 2v_2 \mu_2 w_{22} g_2 +$$
$$+\mu_1^2 w_{11}^2 g_1^2 + 2\mu_1 w_{11} g_1 \mu_2 w_{22} g_2 + \mu_2^2 w_{22}^2 g_2^2 - 4\mu_1 \mu_2 w_{11} w_{22} g_1 g_2 + 4\mu_1 w_{11} g_1 v_2 +$$

$$+4\mu_2 w_{22} g_2 v_1 + 4\mu_1\mu_2 w_{12} w_{21} g_1 g_2 - 4v_1 v_2 =$$

$$= (-v_1 + v_2 + \mu_1 w_{11} g_1)^2 + 2w_{22} g_2(\mu_2(v_1 - v_2 - \mu_1 w_{11} g_1) + 2\mu_1 w_{12} w_{21} g_1) + \mu_2^2 w_{22}^2 g_2^2.$$

Example 3. Consider $\mu_1 = \mu_2 = 10$, $v_1 = v_2 = 1$ and $\theta_1 = 1.2$, $\theta_2 = -0.7$. The regulatory matrix is

$$W = \begin{pmatrix} 0.5 & 2 \\ -2 & 0.5 \end{pmatrix}. \tag{9}$$

The characteristic equation for the critical point $(0.47; 0.47)$ is (6), where $B = -0.48$, $C = 24.66$.

Solving the equation we have $\lambda_1 = 0.2474 - 4.96i$ and $\lambda_2 = 0.2474 + 4.96i$. The type of the critical point is an unstable focus. The periodic solution emerges.

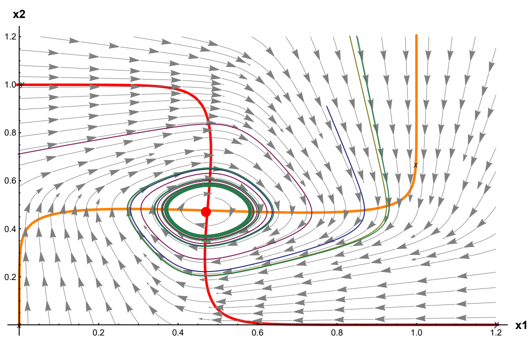

Figure 7: The phase portrait for the system (3) with the regulatory matrix (9). The type of the critical point is an unstable focus.

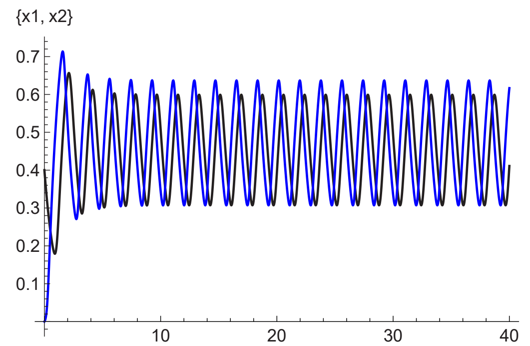

Figure 8: Solutions $(x_1(t), x_2(t))$ for the system (3) with the regulatory matrix (9).

Proposition 4.10. *Two-dimensional system of differential equations* (3) *can have nine critical points if* $w_{11}^2 + w_{22}^2 > 0$.

The respective example is on page 24.

Proposition 4.11. *Suppose that elements w_{11} and w_{22} of the regulatory matrix (4) are zeros. Then the maximal number of equilibria in system (3) is three. Exactly one and exactly two critical points are possible.*

Proposition 4.12. *Suppose that elements w_{11} and w_{22} of the regulatory matrix (4) are not zeros and elements w_{12} and w_{21} are of opposite signs. Then the Hopf bifurcation may occur and the system (3) may have a limit cycle.*

In the case where a stable limit cycle surrounds an unstable critical point, Hopfs bifurcation is called supercritical, and when an unstable limit cycle surrounds a stable critical point, Hopfs bifurcation is called subcritical [21].

Proposition 4.13. *Periodic solutions in system (4) cannot exist if* $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \neq 0$, *where f_1 and f_2 are the right sides of the equations in (4).*

Proof. $w_{12} = \alpha$ and $w_{21} = \beta$ has not closed orbits in Q if $w_{11} = w_{22} = 0$.

$$\frac{\partial\left(\frac{1}{1+e^{-\mu_1(\alpha x_2-\theta_1)}} - v_1 x_1\right)}{\partial x_1} + \frac{\partial\left(\frac{1}{1+e^{-\mu_2(\beta x_1-\theta_2)}} - v_2 x_2\right)}{\partial x_2} = -v_1 - v_2 \neq 0.$$

System (3) cannot have periodic orbits if $w_{11} = w_{22} = 0$, since the Bendixson criterium for non-existence (of periodic orbits) is fulfilled. □

4.3 The behavior of the sigmoid function

Proposition 4.14. *The θ and μ values do not change significantly the behavior of the sigmoid function.*

The case of $w_{12} > 0$, $w_{21} > 0$ is attributed to *activation*, the case of $w_{12} < 0$, $w_{21} < 0$ is interpreted as *inhibition*. Attractors in these cases typically are stable equilibria of the type nodes. If $w_{12}w_{21} < 0$, attracting sets in the form of stable focuses can appear. Even limit cycles are possible, but only if $w_{11}^2 + w_{22}^2 > 0$.

Proposition 4.15. *The parameters w_{12} and w_{21} change the behavior of the sigmoid functions from activation to inhibition.*

Changing w_{12} and w_{21} to the opposite numbers, the functions' behavior changes from activation-activation to inhibition-inhibition. Let us consider $\mu = 2$, $\theta = 0.1$ and we will change the entries of matrix W.

- In Figure 9 the entries of matrix W are $w_{11} = 4, w_{12} = 2, w_{21} = 3, w_{22} = 5$. It is an activation - activation.
- In Figure 10 the entries of matrix W are $w_{11} = -4, w_{12} = 2, w_{21} = 3, w_{22} = 5$. It is an activation - activation.
- In Figure 11 the entries of matrix W are $w_{11} = 4, w_{12} = -2, w_{21} = 3, w_{22} = 5$. It is an inhibition - activation.
- In Figure 12 the entries of matrix W are $w_{11} = 4, w_{12} = 2, w_{21} = -3, w_{22} = 5$. It is an activation - inhibition.
- In Figure 13 the entries of matrix W are $w_{11} = 4, w_{12} = 2, w_{21} = 3, w_{22} = -5$. It is an activation - activation.

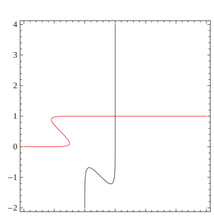

Figure 9:

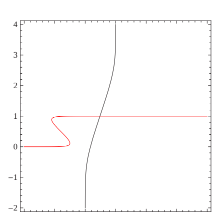

Figure 10:

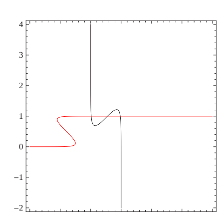

Figure 11:

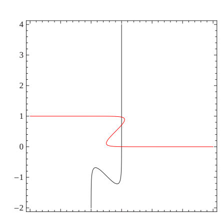

Figure 12:

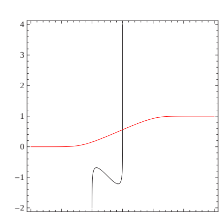

Figure 13:

4.3.1 Activation-activation

Consider $\mu_1 = \mu_2 = 2$, $v_1 = v_2 = 1$ and $\theta_1 = 0.3$, $\theta_2 = 0.5$. The regulatory matrix is

$$W = \begin{pmatrix} 0 & 8 \\ 10 & 0 \end{pmatrix}. \tag{10}$$

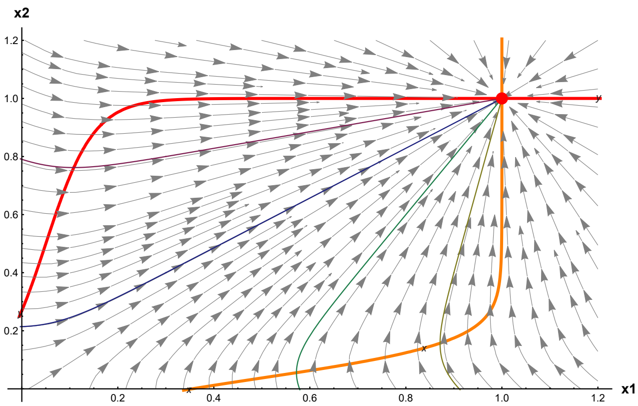

Figure 14: The phase portrait for the system (3) with the regulatory matrix (10). One critical point.

Proposition 4.16. *Three critical points are possible in the system* (3).

Consider $\mu_1 = \mu_2 = 5$, $v_1 = v_2 = 1$ and $\theta_1 = \theta_2 = 3$. The regulatory matrix is

$$W = \begin{pmatrix} 0 & 8 \\ 10 & 0 \end{pmatrix}. \tag{11}$$

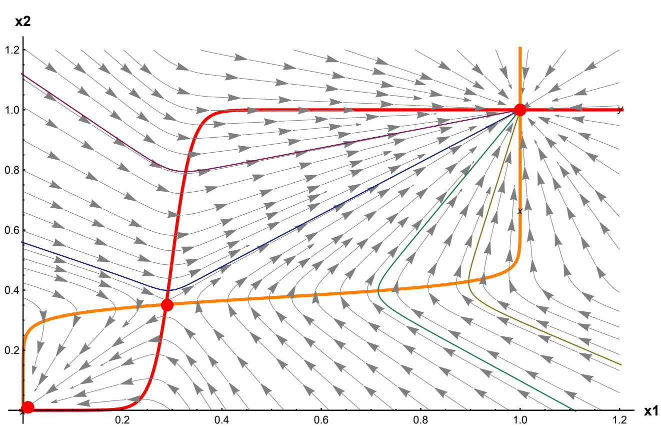

Figure 15: The phase portrait for the system (3) with the regulatory matrix (11). Three critical points.

The types of critical points in Figure 15 are: the stable node, the saddle and the stable node.

Proposition 4.17. *Five critical points are possible in the system* (3).

Consider $\mu = 40$, $v_1 = v_2 = 1$ and $\theta_1 = \theta_2 = 2.5$. The regulatory matrix is

$$W = \begin{pmatrix} 5 & 3 \\ 2 & 3 \end{pmatrix}. \tag{12}$$

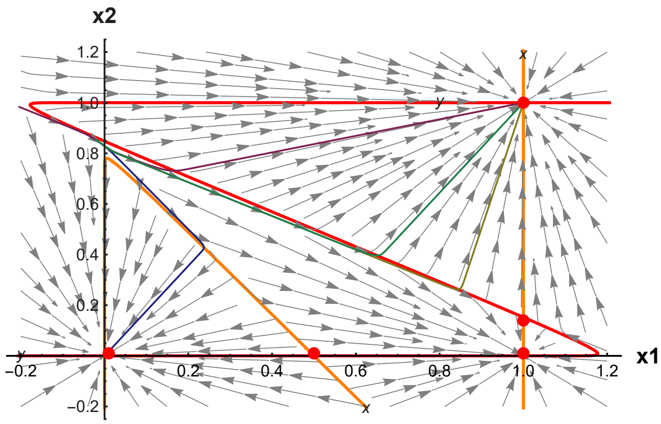

Figure 16: The phase portrait for the system (3) with the regulatory matrix (12). Five critical points.

The types of critical points in Figure 16 are: 3 stable nodes and 2 saddles.

Proposition 4.18. *Seven critical points are possible in the system* (3)*.*

Consider $\mu = 40, v_1 = v_2 = 1$ and $\theta_1 = \theta_2 = 2.5$. The regulatory matrix is

$$W = \begin{pmatrix} 5 & 2.5 \\ 2 & 3 \end{pmatrix}. \tag{13}$$

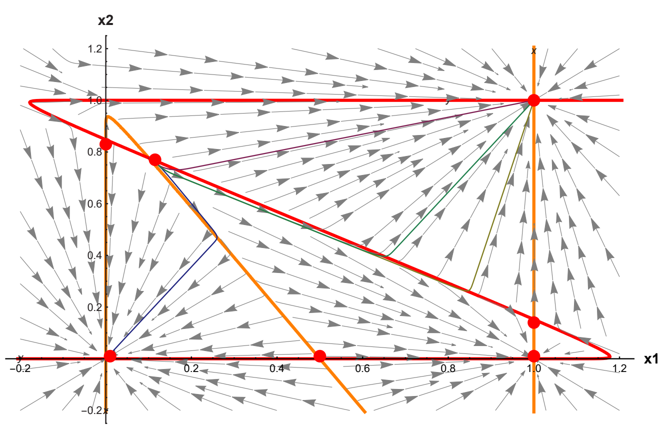

Figure 17: The phase portrait for the system (3) with the regulatory matrix (13). Seven critical points.

The types of critical points in Figure 17 are: 3 stable nodes and 4 saddles.

Proposition 4.19. *In the system* (3) *the maximal number of critical points is nine.*

Consider $\mu = 40$, $v_1 = v_2 = 1$ and $\theta_1 = \theta_2 = 2.5$. The regulatory matrix is

$$W = \begin{pmatrix} 5 & 2.2 \\ 2 & 3 \end{pmatrix}. \tag{14}$$

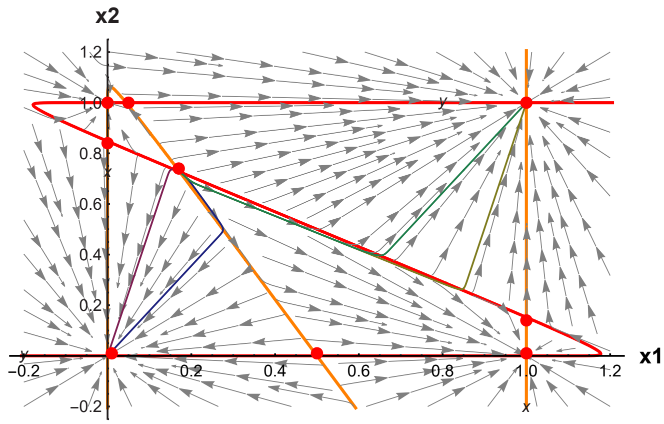

Figure 18: The phase portrait for the system (3) with the regulatory matrix (14). Nine critical points.

The types of critical points in Figure 18 are: 4 stable nodes and 5 saddles.

4.3.2 Activation-inhibition

Proposition 4.20. *The type of critical point in the system* (3) *is the stable focus.*

Consider $\mu_1 = \mu_2 = 2, v_1 = v_2 = 1$ and $\theta_1 = 2, \theta_2 = -1.5$. The regulatory matrix is

$$W = \begin{pmatrix} 0 & 3 \\ -7 & 0 \end{pmatrix}. \tag{15}$$

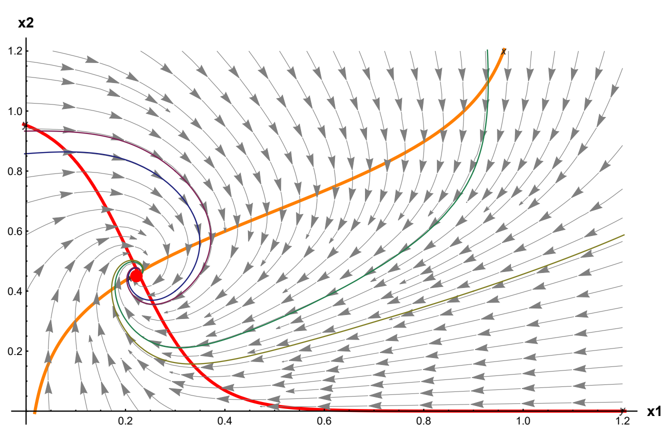

Figure 19: The phase portrait for the system (3) with the regulatory matrix (15). The type of the critical point is a stable focus.

Consider $\mu_1 = \mu_2 = 40,\ v_1 = v_2 = 1$ and $\theta_1 = 3, \theta_2 = 1$. The regulatory matrix is

$$W = \begin{pmatrix} 5 & 2.2 \\ -2 & 3 \end{pmatrix}. \tag{16}$$

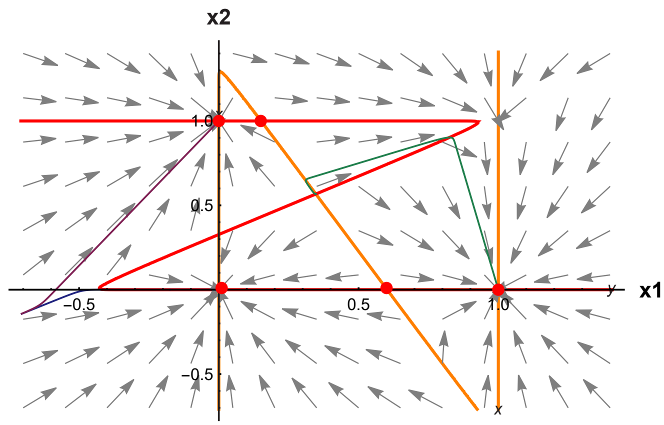

Figure 20: The phase portrait for the system (3) with the regulatory matrix (16).

The types of critical points in Figure 20 are: 3 stable nodes and 2 saddles.

4.3.3 Inhibition-activation

Consider $\mu = 2, v_1 = v_2 = 1$ and $\theta_1 = \theta_2 = 0.1$. The regulatory matrix is

$$W = \begin{pmatrix} 0 & -8 \\ 10 & 0 \end{pmatrix}. \tag{17}$$

In the system (3) with the regulatory matrix (17) there exists only one critical point.

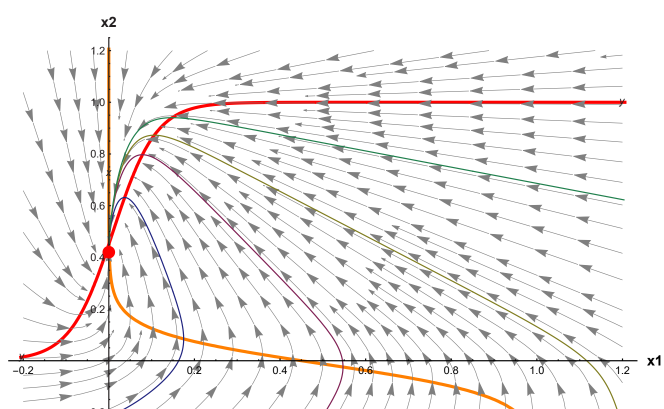

Figure 21: The phase portrait for the system (3) with the regulatory matrix (17). The type of the critical point is a stable focus.

The type is the stable focus.

Consider $\mu_1 = \mu_2 = 40, \upsilon_1 = \upsilon_2 = 1$ and $\theta_1 = 0.5$, $\theta_2 = 0.1$. The regulatory matrix is

$$W = \begin{pmatrix} 6 & -5 \\ 2 & 0 \end{pmatrix}. \tag{18}$$

The types of critical points in Figure 22 are: 2 stable nodes and one saddle.

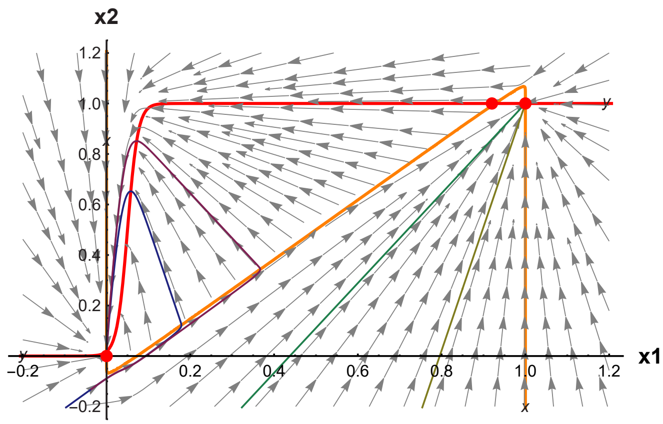

Figure 22: The phase portrait for the system (3) with the regulatory matrix (18).

4.4 Conclusions for two-dimensional systems

The following are true for the system (3):

- there is always an equilibrium state;
- the maximum number of equilibrium states (critical points), except for degenerate cases, is nine;
- the structure of the set of critical points with their maximum number is the same for all considered cases - four stable nodes, four saddles, and an unstable node in center;
- any number of critical points from one to nine is possible;
- periodic solution of the system is possible;
- it is possible to control the system by changing the parameter values.

5 Three-dimensional (3D) systems

Let us consider the system

$$\begin{cases} \dfrac{dx_1}{dt} = \dfrac{1}{1+e^{-\mu_1(w_{11}x_1+w_{12}x_2+w_{13}x_n-\theta_1)}} - v_1x_1, \\ \\ \dfrac{dx_2}{dt} = \dfrac{1}{1+e^{-\mu_2(w_{21}x_1+w_{22}x_2+w_{23}x_n-\theta_2)}} - v_2x_2, \\ \\ \dfrac{dx_3}{dt} = \dfrac{1}{1+e^{-\mu_3(w_{31}x_1+w_{32}x_2+w_{33}x_3-\theta_3)}} - v_3x_3, \end{cases} \tag{19}$$

where μ_i, θ_i and v_i are the parameters, w_{ij} are the coefficients of the so-called regulatory matrix

$$W = \begin{pmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{pmatrix}. \tag{20}$$

The nullclines and the critical points for the system are defined by the relations

$$\begin{cases} x_1 = \dfrac{1}{v_1}\dfrac{1}{1+e^{-\mu_1\,(w_{11}x_1+w_{12}x_2+w_{13}x_3-\theta_1)}}, \\ \\ x_2 = \dfrac{1}{v_2}\dfrac{1}{1+e^{-\mu_2\,(w_{21}x_1-w_{22}x_2+w_{23}x_3-\theta_2)}}, \\ \\ x_3 = \dfrac{1}{v_3}\dfrac{1}{1+e^{-\mu_2\,(w_{21}x_1+w_{22}x_2+w_{33}x_3-\theta_3)}}. \end{cases}$$

5.1 Linearized system

The linearized system for any critical point (x_1^*, x_2^*, x_3^*) is

$$\begin{cases} u_1' = -v_1u_1 + \mu_1w_{11}g_1u_1 + \mu_1w_{12}g_1u_2 + \mu_1w_{13}g_1u_3, \\ u_2' = -v_2u_2 + \mu_2w_{21}g_2u_1 + \mu_2w_{22}g_2u_2 + \mu_2w_{23}g_2u_3, \\ u_3' = -v_3u_3 + \mu_3w_{31}g_3u_1 + \mu_3w_{32}g_3u_2 + \mu_3w_{33}g_3u_3, \end{cases}$$

where

$$g_1 = \frac{e^{-\mu_1(w_{11}x_1^*+w_{12}x_2^*+w_{13}x_3^*-\theta_1)}}{[1+e^{-\mu_1(w_{11}x_1^*+w_{12}x_2^*+w_{13}x_3^*-\theta_1)}]^2}, \tag{21}$$

$$g_2 = \frac{e^{-\mu_2(w_{21}x_1^*+w_{22}x_2^*+w_{23}x_3^*-\theta_2)}}{[1+e^{-\mu_2(w_{21}x_1^*+w_{22}x_2^*+w_{23}x_3^*-\theta_2)}]^2}, \tag{22}$$

$$g_3 = \frac{e^{-\mu_3(w_{31}x_1^*+w_{32}x_2^*+w_{33}x_3^*-\theta_3)}}{[1+e^{-\mu_3(w_{31}x_1^*+w_{32}x_2^*+w_{33}x_3^*-\theta_3)}]^2}. \tag{23}$$

One has

$$A - \lambda I = \begin{vmatrix} \mu_1w_{11}g_1 - v_1 - \lambda & \mu_1w_{12}g_1 & \mu_1w_{13}g_1 \\ \mu_2w_{21}g_2 & \mu_2w_{22}g_2 - v_2 - \lambda & \mu_2w_{23}g_2 \\ \mu_3w_{31}g_3 & \mu_3w_{32}g_3 & \mu_3w_{33}g_3 - v_3 - \lambda \end{vmatrix}$$

and the characteristic equation is

$$\det|A-\lambda I| = -\lambda^3 + \lambda^2(-v_1 - v_2 - v_3 + \mu_1 w_{11} g_1 + \mu_2 w_{22} g_2 + \mu_3 w_{33} g_3) + \lambda(g_1 v_3 \mu_1 w_{11} +$$

$$+\mu_2 w_{22} g_2 v_3 + g_1 g_2 w_{21} \mu_1 \mu_2 w_{12} - g_1 g_2 w_{11} w_{22} \mu_1 \mu_2 + g_1 g_3 w_{31} w_{13} \mu_1 \mu_3 -$$

$$-g_1 g_3 w_{11} w_{33} \mu_1 \mu_3 + g_2 g_3 w_{32} w_{23} \mu_2 \mu_3 - g_2 g_3 w_{22} w_{33} \mu_2 \mu_3 - v_1(v_2 + v_3 - g_2 w_{22} \mu_2 - g_3 w_{33} \mu_3) +$$

$$+v_2(-v_3 + g_1 w_{11} \mu_1 + g_3 w_{33} \mu_3)) + v_1(v_2(-v_3 + g_3 w_{33} \mu_3) + g_2 \mu_2(v_3 w_{22} + g_3 w_{32} w_{23} \mu_3 - g_3 w_{22} w_{33} \mu_3)) +$$

$$+g_1 \mu_3(v_2(v_3 w_{11} + g_3(w_{31} w_{13} - w_{11} w_{33})\mu_3) + g_2 \mu_2(v_3(w_{21} w_{12} - w_{11} w_{22}) +$$

$$+g_3(-w_{31} w_{22} w_{13} + w_{21} w_{32}\, w_{13} + w_{31} w_{12} w_{23} - w_{11} w_{32}\, w_{23} - w_{21} w_{12} w_{33} + w_{11} w_{22}\, w_{33})\mu_3)) = 0.$$

The characteristic equation can be rewritten as

$$-\lambda^3 + A\lambda^2 + B\lambda + C = 0, \tag{24}$$

where

$$A = -(v_1 + v_2 + v_3) + g_1 w_{11} \mu_1 + g_2 w_{22} \mu_2 + g_3 w_{33} \mu_3,$$

$$\begin{aligned} B &= \mu_1 \mu_2 w_{31} w_{13} g_1 g_3 - \mu_2 \mu_3 w_{32} w_{23} g_2 g_3 + \mu_1 \mu_2 w_{21} w_{12} g_1 g_2 \\ &-(\mu_2 w_{22} g_2 - v_2)(\mu_3 w_{33} g_3 - v_3) - (\mu_1 w_{11} g_1 - v_1)(\mu_3 w_{33} g_3 - v_3) \\ &-(\mu_1 w_{11} g_1 - v_1)(\mu_2 w_{22} g_2 - v_2), \end{aligned}$$

$$\begin{aligned} C &= (\mu_1 w_{11} g_1 - v_1)(\mu_2 w_{22} g_2 - v_2)(\mu_3 w_{33} g_3 - v_3) + \mu_1 \mu_2 \mu_3 w_{21} w_{32} w_{23} g_1 g_2 g_3 \\ &+\mu_1 \mu_2 \mu_3 w_{31} w_{12} w_{23} g_1 g_2 g_3 - \mu_1 \mu_3 w_{31} w_{13} g_1 g_3 (\mu_2 w_{22} g_2 - v_2) \\ &-\mu_2 \mu_3 w_{32} w_{23} g_2 g_3 (\mu_1 w_{11} g_1 - v_1) - \mu_1 \mu_2 w_{21} w_{12} g_1 g_2 (\mu_3 w_{33} g_3 - v_3). \end{aligned}$$

5.1.1 Facts

Proposition 5.1. *The vector field $(f_1(x_1,x_2,x_3), f_2(x_1,x_2,x_3), f_3(x_1,x_2,x_3))$, where f_1, f_2 and f_3 are the right sides of the equations in (19), is directed inward on the boundary of the domain $Q_3 := \{(x_1,x_2,x_3) : 0 < x_1 < \frac{1}{v_1}, 0 < x_2 < \frac{1}{v_2}, 0 < x_3 < \frac{1}{v_3}\}$.*

Proof. Take one of faces of the parallelepiped Q_3, for example, $x_1 = 0$. The vector field there in the x_1 direction is $f_1 - v_1$, $x_1 = f > 0$. Take face $x_1 = \dfrac{1}{v_1}$. The vector field there in the x_1 direction is $f_1 - v_1$, $x_1 = f_1 - v_1$, $\dfrac{1}{v_1} = f_1 - 1 < 0$. In both cases, the vector field along the x_1 axis is directed inside Q_3. Similarly, other faces of Q_3 can be considered.□

Proposition 5.2. *System (19) has at least one equilibrium (critical point). All equilibria are located in the open box $Q_3 := \{(x_1,x_2,x_3) : 0 < x_1 < \frac{1}{v_1}, 0 < x_2 < \frac{1}{v_2}, 0 < x_3 < \frac{1}{v_3}\}$.*

5.2 Critical points

The three-dimensional system has three eigenvalues. Two main possibilities exist: either the three eigenvalues are real or two of them are complex conjugates. A critical point is stable if all eigenvalues have negative real parts; it is unstable if at least one eigenvalue has positive real part.

- **Node**. All eigenvalues are real and have the same sign. The node is stable (unstable) when the eigenvalues are negative (positive) [97].
- **Saddle**. All eigenvalues are real and at least one of them is positive and at least one is negative. Saddles are always unstable [97].
- **Focus – Node**. It has one real eigenvalue and a pair of complex-conjugate eigenvalues, and all eigenvalues have real parts of the same sign. The critical point is stable (unstable) when the sign is negative (positive) [97].
- **Saddle – Focus**. Negative real eigenvalue and complex eigenvalues with positive real part (unstable focus), and positive real eigenvalue and complex eigenvalues with negative real part (stable focus). This type of critical point is unstable [46].

5.3 Particular cases

5.3.1 Case 1

Let $v_1 = v_2 = v_3 = 1$.

We have

$$\begin{aligned}
&\det|A-\lambda I| = -\Lambda^3 + (\mu_1 w_{11} g_1 + \mu_2 w_{22} g_2 + \mu_3 w_{33} g_3)\Lambda^2 \\
&+[\mu_1\mu_3 g_1 g_3(w_{31}w_{13} - w_{11}w_{33}) + \mu_2\mu_3 g_2 g_3(w_{32}w_{23} - w_{22}w_{33}) \\
&+\mu_1\mu_2 g_1 g_2(w_{21}w_{12} - w_{11}w_{22})]\Lambda \\
&-\mu_1\mu_2\mu_3 g_1 g_2 g_3(w_{11}w_{32}w_{23} + w_{21}w_{12}w_{33} + w_{31}w_{22}w_{13} \\
&-w_{11}w_{22}w_{33} - w_{12}w_{23}w_{31} - w_{13}w_{21}w_{32}) = 0,
\end{aligned}$$

where $\Lambda = \lambda + 1$.

The characteristic equation is (24), where

$$A = -3 + g_1 w_{11}\mu_1 + g_2 w_{22}\mu_2 + g_3 w_{33}\mu_3,$$

$$\begin{aligned}
B = {}&\mu_1\mu_2 w_{12}w_{21}g_1g_2 - (-1 + g_1w_{11}\mu_1)(-1 + g_2w_{22}\mu_2) + \mu_1\mu_3w_{13}w_{31}g_1g_3 + \mu_2\mu_3w_{23}w_{32}g_2g_3 \\
&-(-1 + g_1w_{11}\mu_1)(-1 + g_3w_{33}\mu_3) - (-1 + g_2w_{22}\mu_2)(-1 + g_3w_{33}\mu_3),
\end{aligned}$$

$$\begin{aligned}
C = {}&g_1g_2g_3w_{12}w_{23}w_{31}\mu_1\mu_2\mu_3 + g_1g_2g_3w_{13}w_{21}w_{32}\mu_1\mu_2\mu_3 - g_2g_3w_{23}w_{32}(-1 + g_1w_{11}\mu_1)\mu_2\mu_3 \\
&- g_1g_3w_{13}w_{31}\mu_1(-1 + g_2w_{22}\mu_2)\mu_3 - g_1g_2w_{12}w_{21}\mu_1\mu_2 - (-1 + g_3w_{33}\mu_3) \\
&+ (-1 + g_1w_{11}\mu_1)(-1 + g_2w_{22}\mu_2)(-1 + g_3w_{33}\mu_3).
\end{aligned}$$

5.3.2 Case 2

Let $w_{11} = w_{22} = w_{33} = 0$. The regulatory matrix is

$$W = \begin{pmatrix} 0 & w_{12} & w_{13} \\ w_{21} & 0 & w_{23} \\ w_{31} & w_{32} & 0 \end{pmatrix} \tag{25}$$

and the system of differential equations takes the form

$$\begin{cases} x_1' = \dfrac{1}{1 + e^{-\mu_1(w_{12}x_2 + w_{13}x_3 - \theta_1)}} - v_1 x_1, \\ x_2' = \dfrac{1}{1 + e^{-\mu_2(w_{21}x_1 + w_{23}x_3 - \theta_2)}} - v_2 x_2, \\ x_3' = \dfrac{1}{1 + e^{-\mu_3(w_{31}x_1 + w_{32}x_2 - \theta_3)}} - v_3 x_3. \end{cases}$$

The linearized system for a critical point (x_1^*, x_2^*, x_3^*) is then

$$\begin{cases} u_1' = -v_1 u_1 + \mu_1 w_{12} g_1 u_2 + \mu_1 w_{13} g_1 u_3, \\ u_2' = -v_2 u_2 + \mu_2 w_{21} g_2 u_1 + \mu_2 w_{23} g_2 u_3, \\ u_3' = -v_3 u_3 + \mu_3 w_{31} g_3 u_1 + \mu_3 w_{32} g_3 u_2, \end{cases}$$

where g_1, g_2, g_3, given in (21) to (23), are adapted to the case of the regulatory matrix (25).

The characteristic equation is

$$\det|A - \lambda I| = -\lambda^3 + \lambda(g_2 g_3 w_{32} w_{23} \mu_2 \mu_3 + g_1 \mu_1 (g_2 w_{21} w_{12} \mu_2 + g_3 w_{31} w_{13} \mu_3)) +$$

$$+ g_1 g_2 g_3 (w_{21} w_{32} w_{13} + w_{31} w_{12} w_{23}) \mu_1 \mu_2 \mu_3 = 0.$$

The characteristic equation can be rewritten as (24), where

$$A = -(v_1 + v_2 + v_3),$$

$$B = -v_1 v_2 - v_1 v_3 - v_2 v_3 + g_1 g_2 w_{12} w_{21} \mu_1 \mu_2 + g_1 g_3 w_{13} w_{31} \mu_1 \mu_3 + g_2 g_3 w_{23} w_{32} \mu_2 \mu_3,$$

$$C = -v_1 v_2 v_3 + g_1 g_2 v_3 w_{12} w_{21} \mu_1 \mu_2 + g_1 g_3 v_2 w_{13} w_{31} \mu_1 \mu_3 + g_2 g_3 v_1 w_{23} w_{32} \mu_2 \mu_3 + g_1 g_2 g_3 w_{23} w_{32} \mu_2 \mu_3$$
$$+ g_1 g_2 g_3 w_{12} w_{23} w_{31} \mu_1 \mu_2 \mu_3 + g_1 g_2 g_3 w_{13} w_{21} w_{32} \mu_1 \mu_2 \mu_3.$$

5.3.3 Case 3

Let $v_1 = v_2 = v_3 = 1$, $w_{11} = w_{22} = w_{33} = 0$. The regulatory matrix is

$$W = \begin{pmatrix} 0 & w_{12} & w_{13} \\ w_{21} & 0 & w_{23} \\ w_{31} & w_{32} & 0 \end{pmatrix} \tag{26}$$

and the system of differential equations takes the form

$$\begin{cases} x_1' = \dfrac{1}{1+e^{-\mu_1(w_{12}x_2+w_{13}x_3-\theta_1)}} - x_1, \\ x_2' = \dfrac{1}{1+e^{-\mu_2(w_{21}x_1+w_{23}x_3-\theta_2)}} - x_2, \\ x_3' = \dfrac{1}{1+e^{-\mu_3(w_{31}x_1+w_{32}x_2-\theta_3)}} - x_3. \end{cases}$$

The linearized system for a critical point (x_1^*, x_2^*, x_3^*) is then

$$\begin{cases} u_1' = -u_1 + \mu_1 w_{12} g_1 u_2 + \mu_1 w_{13} g_1 u_3, \\ u_2' = -u_2 + \mu_2 w_{21} g_2 u_1 + \mu_2 w_{23} g_2 u_3, \\ u_3' = -u_3 + \mu_3 w_{31} g_3 u_1 + \mu_3 w_{32} g_3 u_2, \end{cases}$$

where g_1, g_2, g_3, given in (21) to (23), are adapted to the case of the regulatory matrix (26). The characteristic equation is

$$-\Lambda^3 + B\Lambda + C = 0,$$

where $\Lambda = \lambda + 1$,

$$B = \mu_1\mu_3 g_1 g_3 (w_{31}w_{13}) + \mu_2\mu_3 g_2 g_3 (w_{32}w_{23}) + \mu_1\mu_2 g_1 g_2 (w_{21}w_{12}),$$

$$C = \mu_1\mu_2\mu_3 g_1 g_2 g_3 (w_{12}w_{23}w_{31} + w_{13}w_{21}w_{32}).$$

5.4 Cardano formulas

Gerolamo Cardano (1501-1576) was an Italian mathematician, engineer, philosopher, physician, astrologer. He published fundamental works on algebra, probability theory, and mechanics, which had a huge impact on the development of science.

For further analysis let us recall the Cardano formulas applied to the equation

$$y^3 + py + q = 0. \tag{27}$$

It has complex roots if

$$Q := \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2$$

is positive. The complex roots are given by expressions

$$y_{2,3} = -\frac{a+b}{2} \pm i(a-b)\frac{\sqrt{3}}{2},$$

where

$$a = \left(-\frac{q}{2} + \sqrt{Q}\right)^{\frac{1}{3}}, \quad b = \left(-\frac{q}{2} - \sqrt{Q}\right)^{\frac{1}{3}}$$

are real cubic roots satisfying $a \cdot b = -\frac{p}{3}$. The remaining real root of equation (27) $y_1 = a + b$ is real.

5.4.1 Case 1

The regulatory matrix is

$$W = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

The system is

$$\begin{cases} x_1' = \dfrac{1}{1+e^{-\mu_1(x_2-\theta_1)}} - v_1 x_1, \\ x_2' = \dfrac{1}{1+e^{-\mu_2(x_1-x_3-\theta_2)}} - v_2 x_2, \\ x_3' = \dfrac{1}{1+e^{-\mu_3(-x_2-\theta_3)}} - v_3 x_3. \end{cases}$$

The linearized system for a critical point (x_1^*, x_2^*, x_3^*) is then $(v_1 = v_2 = v_3 = 1)$

$$\begin{cases} u_1' = -u_1 + \mu_1 g_1 u_2, \\ u_2' = -u_2 + \mu_2 g_2 u_1 - \mu_2 g_2 u_3, \\ u_3' = -u_3 - \mu_3 g_3 u_2, \end{cases}$$

where

$$g_1 = \frac{e^{-\mu_1(x_2^*-\theta_1)}}{[1+e^{-\mu_1(x_2^*-\theta_1)}]^2},$$

$$g_2 = \frac{e^{-\mu_2(x_1^*-x_3^*-\theta_2)}}{[1+e^{-\mu_2(x_1^*-x_3^*-\theta_2)}]^2},$$

$$g_3 = \frac{e^{-\mu_3(-x_2^*-\theta_3)}}{[1+e^{-\mu_3(-x_2^*-\theta_3)}]^2}.$$

The characteristic equation is

$$-\Lambda^3 + B\Lambda + C = 0, \tag{28}$$

where $\Lambda = \lambda + 1$,

$$B = \mu_2\mu_3 g_2 g_3 + \mu_1\mu_2 g_1 g_2 > 0,$$

$$C = 0.$$

The equation (28) has three roots

$$\Lambda_1 = 0, \quad \Lambda_{2,3} = \pm\sqrt{B}$$

and, consequently,

$$\lambda_1 = -1, \quad \lambda_{2,3} = -1 \pm \sqrt{B}.$$

Proposition 5.3. *If $B > 1$, then the respective critical point is a 3D-saddle; if $B < 1$, then the critical point is a stable node.*

5.4.2 Case 2

The regulatory matrix is

$$W = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The system takes the form

$$\begin{cases} x_1' = \dfrac{1}{1 + e^{-\mu_1(-x_2-\theta_1)}} - v_1 x_1, \\ x_2' = \dfrac{1}{1 + e^{-\mu_2(x_1-x_3-\theta_2)}} - v_2 x_2, \\ x_3' = \dfrac{1}{1 + e^{-\mu_3(x_2-\theta_3)}} - v_3 x_3. \end{cases} \tag{29}$$

The linearized system for a critical point (x_1^*, x_2^*, x_3^*) is $(v_1 = v_2 = v_3 = 1)$

$$\begin{cases} u_1' = -u_1 - \mu_1 g_1 u_2, \\ u_2' = -u_2 + \mu_2 g_2 u_1 - \mu_2 g_2 u_3, \\ u_3' = -u_3 + \mu_3 g_3 u_2, \end{cases}$$

where

$$g_1 = \frac{e^{-\mu_1(-x_2^*-\theta_1)}}{[1 + e^{-\mu_1(x_2^*-\theta_1)}]^2},$$

$$g_2 = \frac{e^{-\mu_2(x_1^*-x_3^*-\theta_2)}}{[1 + e^{-\mu_2(x_1^*-x_3^*-\theta_2)}]^2},$$

$$g_3 = \frac{e^{-\mu_3(x_2^*-\theta_3)}}{[1 + e^{-\mu_3(-x_2^*-\theta_3)}]^2}.$$

The characteristic equation is

$$-\Lambda^3 + B\Lambda + C = 0, \tag{30}$$

where $\Lambda = \lambda + 1$,

$$B = -\mu_2\mu_3 g_2 g_3 - \mu_1\mu_2 g_1 g_2,$$

$$C = 0.$$

The equation (30) has three roots

$$\Lambda_1 = 0, \quad \Lambda_{2,3} = \pm\sqrt{-B}i$$

and, consequently,

$$\lambda_1 = -1, \quad \lambda_{2,3} = -1 \pm \sqrt{-B}i.$$

Proposition 5.4. *A unique critical point of the system (29), where $v_i = 1$, $i = 1, 2, 3$, is a stable focus-node.*

Proof. Let us prove that a critical point is unique. Write the system of nullclines as

$$\begin{cases} x_1 = \dfrac{1}{1+e^{-\mu_1(-x_2-\theta_1)}} := f_1(-x_2), \\ x_2 = \dfrac{1}{1+e^{-\mu_2(x_1-x_3-\theta_2)}} := f_2(x_1, -x_3), \\ x_3 = \dfrac{1}{1+e^{-\mu_3(x_2-\theta_3)}} := f_3(x_2), \end{cases} \tag{31}$$

where f_i are evident sigmoid functions. Any critical point is a solution of the system (31). We can write the second equation as

$$x_2 = f_2(f_1(-x_2), -f_3(x_2)), \tag{32}$$

where $f_2(\eta, \xi) = \dfrac{1}{1+e^{-\mu_2(\eta+\xi-\theta_2)}}$. It is increasing function of x_2 on the left in (32) and decreasing function of x_2 on the right. The graphs of both functions can intersect only once. □

5.5 Chaos

Under chaos in ancient Greek mythology understood the pre-life confusion. Greek "chaos" is the infinite first everyday mass, which subsequently gave rise to all the existing. Physicists call this science - "nonlinear dynamics", mathematicians - "chaos theory", all the rest - "nonlinear science".

Chaos is a multifaceted phenomenon that is not easily classified or identified. There is no universally accepted definition for chaos, but the following characteristics are nearly always displayed by the solutions of chaotic systems [39].

Characteristics of chaos

- A characteristic of chaotic behavior is the existence of an attractor to which all sufficiently nearby solutions converge, given sufficient time [23].
- A typical characteristic of chaotic solutions is the geometric form of the attractors. The attractors typically are twisted and 'strange', meaning that they have fractional (fractal) dimension, although this is not necessarily the case [23].
- Sensitivity to initial conditions [39].

Definition 5.1. *A chaotic system is a deterministic system that exhibits irregular and unpredictable behavior [47].*

Research on chaotic systems had a practical effect since Edward Norton Lorenz established chaos theory in 1963. Chaos should be expected to be a very common basic dynamical state in a variety of systems. Chaotic dynamics is very important in different fields such as robotics, economics, cryptography, chemistry, medicine (studying epilepsy to predict seizures, taking into account the initial state of the organism) and biology (in the study of uneven heart rate and an uneven number of diseases) [49].

Proposition 5.5. *In dynamical systems that include three or more equations, there may be even more unusual attractors, which are commonly called strange or chaotic attractors.*

Floris Takens (1940 - 2010) a Dutch mathematician known for contributions to the theory of differential equations, the theory of dynamical systems, chaos theory and fluid mechanics. Introduced the concept of a "strange attractor". He was the first to show how chaotic attractors could be learned by neural networks [7].

Proposition 5.6. *It is possible to find a chaotic attractor in differential systems presenting chaotic behavior [55].*

Definition 5.2. *A strange attractor, (chaotic attractor, fractal attractor) is an attractor that exhibits sensitivity to initial conditions [39].*

Definition 5.3. *A fractal is an object that displays self-similarity under magnification and can be constructed using a simple motif (an image repeated on ever-reduced scales) [39].*

Such strange objects were identified in nature. Sunflowers and broccoli (Figure 24), sea shells, fern, snowflakes (Figure 23), mountain chasms, coastlines, lightning bolts (Figure 25), tree branches, river beds, turbulent eddies, human vascular system. These fractal geometries play a significant role in the characterization of chaotic dynamical processes, fractal dimension is therefore an important attribute of such a process [34].

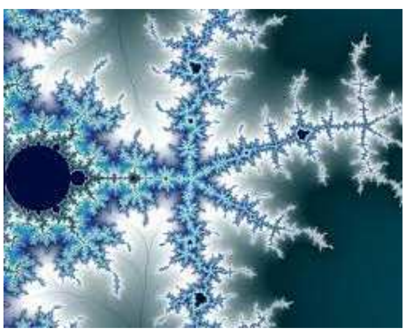

Figure 23: The picture from www.esa.org

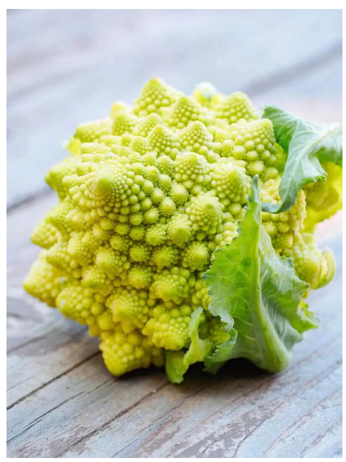

Figure 24: Remarkable Romanesco Broccoli. The picture from www.gardenbetty.com

Figure 25: The picture from www.zmescience.com

The Cantor set constitutes a fractal object. The concept of dimension has to be broadened to include fractal geometries associated with chaotic dynamics [34].

Definition 5.4. *The Cantor set or Cantor's Middle Thirds set, is given by taking the interval $[0, 1]$ (set C_0), removing the open middle third C_1, removing the middle third of each of the two remaining pieces C_2, and continuing this procedure ad infinitum.*

At the kth stage in the Cantor set, there will be $N = 2^k$ segments each of length $l = 3^{-k}$. If this process is continued to infinity, then

$$\lim_{k\to\infty} 2^k = \infty \qquad \text{and} \qquad \lim_{k\to\infty} 3^{-k} = 0. \quad [39]$$

Figure 26: Sequential construction of the Cantor set.

The Koch Snowflake was created by the Swedish mathematician Niels Fabian Helge von Koch in 1904. The Koch Curve is constructed by replacing a unit line segment with a motif consisting of four line segments each of length $\frac{1}{3}$.

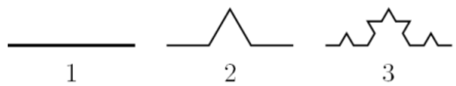

Figure 27: Sequential construction of a Koch snowflake.

At the kth stage in the Koch Curve, there will be $N = 4^k$ segments each of length $l = 3^{-k}$. If this process is continued to infinity, then

$$\lim_{k\to\infty} 4^k = \infty \qquad \text{and} \qquad \lim_{k\to\infty} 3^{-k} = 0. \quad [39]$$

Definition 5.5. *A self-similar fractal has fractal dimension (or Hausdorff index) D_f given by*

$$D_f = \frac{\ln N(l)}{-\ln l},$$

where l represents a scaling and $N(l)$ denotes the number of segments of length l [39].

Definition 5.6. *A fractal is an object that has noninteger fractal dimension. (This is an alternative to Definition*(5.3)*)[39].*

The fractal dimension for the Cantor set is

$$D_f = \frac{\ln 2}{-\ln \frac{1}{3}} = \frac{\ln 2}{\ln 3} \approx 0.6309298$$

The Cantor set is denser than a point, but less dense than a line, because the dimension of a point is zero, but dimension of the line is one.

The fractal dimension for the Koch Curve is

$$D_f = \frac{\ln 4}{-\ln \frac{1}{3}} = \frac{\ln 4}{\ln 3} \approx 1.2618595$$

The Koch Curve is denser than the line, but less dense than a plane, because the dimension of the line is one, but the dimension of the plane is two.

5.6 Lyapunov exponents

Aleksandr Lyapunov (1857 - 1918) is a mathematician and mechanic, academician of the St. Petersburg Academy of Sciences. The most important achievement of the scientist is the creation of a modern theory of the stability of equilibrium and the movement of mechanical systems, determined by a finite number of parameters [98].

The Lyapunov exponents are an important tool for the characterization of an attractor of a finite-dimensional nonlinear dynamic system and their excessive sensitivity to initial conditions [19]. The Lyapunov exponent is an approach to detect chaos, and it is a measure of the speeds at which initially nearby trajectories of the system diverge [47].

Relationships between the Lyapunov exponents and the properties and types of attractors:

1. One-dimensional system. In this case only a stable fixed point can be an attractor. There exists one negative Lyapunov exponent (LE in short) denoted by $LE_1 = (-)$.

2. Two-dimensional system. In 2D systems, there are two types of attractors: stable fixed points and limit cycles. The corresponding LEs follow:

   - $(LE_1, LE_2) = (-, -)$ - stable point;
   - $(LE_1, LE_2) = (0, -)$ - stable limit cycle (one exponent is equal to zero).

3. Three-dimensional system. In 3D phase space, there exist four types of attractors: stable points, limit cycles, 2D tori and strange attractors. The following set of LEs characterizes possible dynamical situations to be met:

   - $(LE_1, LE_2, LE_3) = (-, -, -)$ - stable fixed point;
   - $(LE_1, LE_2, LE_3) = (0, -, -)$ - stable limit cycle;
   - $(LE_1, LE_2, LE_3) = (0, 0, -)$ - stable 2D tori;
   - $(LE_1, LE_2, LE_3) = (+, 0, -)$ - strange attractor.

Symbols $(+, 0, -)$ mean that for the analyzed attractor, there is one direction in a 3D space, where exponential stretching is exhibited, the second direction indicates neutral stability, and the third one-exponential compression [6].

5.6.1 Properties of Lyapunov exponents

1. The number of Lyapunov exponents is equal to the number of phase space dimensions, or the order of the system of differential equations. They are arranged in descending order [79].

2. The largest Lyapunov exponent of a stable system does not exceed zero [47].

3. A chaotic system has at least one positive Lyapunov exponent, and the more positive the largest Lyapunov exponent, the more unpredictable the system is [47].

4. To have a dissipative dynamical system, the values of all Lyapunov exponents should sum to a negative number [79].

5. A hyperchaotic system is defined as a chaotic system with at least two positive Lyapunov exponents. Combined with one null exponent and one negative exponent, the minimal dimension for a hyperchaotic system is four [86].

Proposition 5.7. *Dissipative systems exhibit chaos for most initial conditions in a specified range of parameters. A conservative system exhibits periodic and quasi-periodic solutions for most values of parameters and initial conditions, and can exhibit chaos for special values only [79].*

Proposition 5.8. *Only dissipative dynamical systems have attractors [46].*

Proposition 5.9. *If the Jacobian of the vector field in ODE is negative then the system is a dissipative [94].*

In the thesis for Lyapunov exponents calculation the package "lce.m for Mathematica" was used [99]. Another Wolfram Mathematica program "Lynch-DSAM.nb" was also used to check the correctness of Lyapunov exponents calculation [39].

5.7 Examples

Example 1. The system (19) with the matrix

$$W = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{33}$$

and $\mu_1 = \mu_2 = 7, \mu_3 = 5$, $v_1 = v_2 = v_3 = 1$ and $\theta_1 = 0.8$, $\theta_2 = 1.0$, $\theta_3 = 0.5$ has nine attractive critical points, which can be observed in Figure 28. Six of intersections of red and green with blue surfaces, at the corners of a cube.

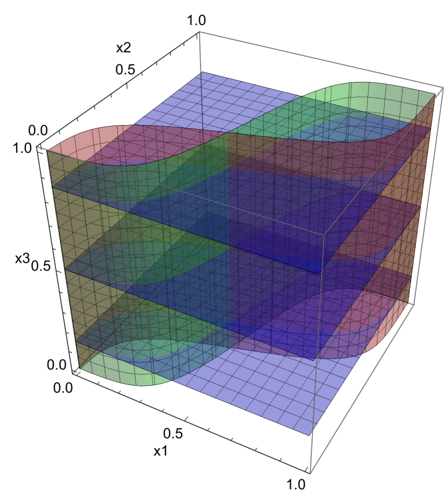

Figure 28: The visualization of nullclines and nine critical points for the system (19) with the regulatory matrix (33).

5.7.1 Periodic solutions

Example 2. The system (19) with the matrix

$$W = \begin{pmatrix} k & 0 & -1 \\ -1 & k & 0 \\ 0 & -1 & k \end{pmatrix}. \tag{34}$$

This matrix contains the inhibitor cycle (elements -1) and auto-activation (elements k). For values k in the interval $(0.36, 2)$, the $3D$ system has a periodic solution that attracts other solutions [59].

The initial conditions are

$$x_1(0) = 0.5;\ x_2(0) = 0.5;\ x_3(0) = 0.35.$$

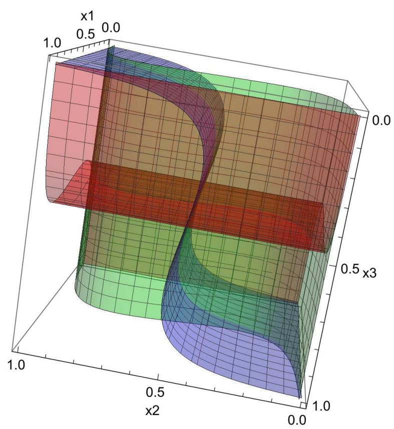

Figure 29: Nullclines for the system (19) with the regulatory matrix (34) x_1 - red, x_2 - green, x_3 - blue, $k = 1$, $\mu_i = 5$, $\theta_i = \frac{k-1}{2}$.

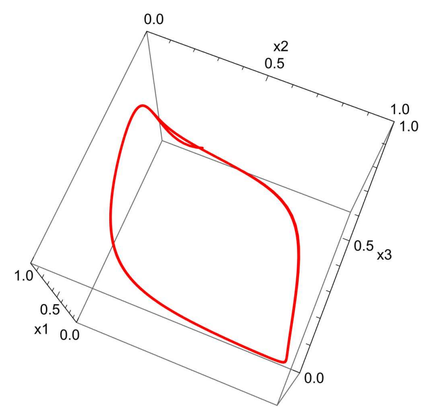

Figure 30: The periodic solution of the system (19) with the regulatory matrix (34), $k = 1$.

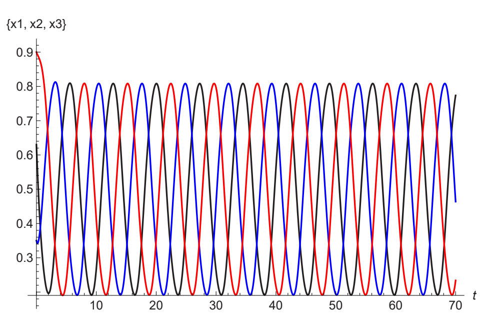

Figure 31: The graphs of $x_i(t), i = 1, 2, 3$ for the system (19) with the regulatory matrix (34).

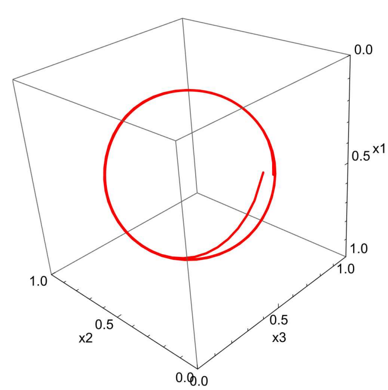

Figure 32: The periodic solution of the system (19) with the regulatory matrix (34), $k = 0.5$.

Example 3. Consider $\mu_1 = 5$, $\mu_2 = 15$, $\mu_3 = 5$, $v_1 = v_2 = v_3 = 1$ and $\theta_1 = 1.2$, $\theta_2 = 0.5$, $\theta_3 = -0.6$. The regulatory matrix of the system (19) is

$$W = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}. \tag{35}$$

The nullclines are depicted in Figure 33. There are exactly three critical points.

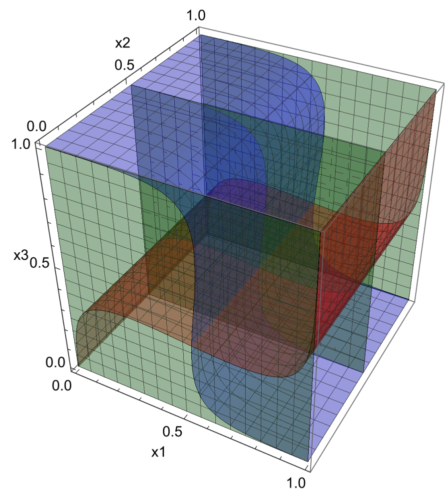

Figure 33: Nullclines x_1 - red, x_2 - green, x_3 - blue of the system (19) with the regulatory matrix (35).

The characteristic equation for critical point $(0.537; 0.001; 0.346)$ is

$$-\lambda^3 + A\lambda^2 + B\lambda + C = 0, \tag{36}$$

where $A = -0.616403$, $B = -5.28938$ and $C = -5.61417$.

Solving the equation we have $\lambda_1 = -0.99$, $\lambda_{2,3} = 0.188 \pm 2.371i$. The type of the critical point is unstable saddle-focus.

The characteristic equation for critical point $(0.537; 0.5; 0.346)$ is (36), where $A = 3.125$, $B = -6.693$ and $C = 15.569$.

Solving the equation we have $\lambda_1 = 2.75$, $\lambda_{2,3} = 0.187 \pm 2.371i$. The type of the critical point is unstable focus-node.

The characteristic equation for critical point $(0.537; 0.99; 0.346)$ is (36), where $A = -0.6164$, $B = -5.289$ and $C = -5.614$.

Solving the equation we have $\lambda_1 = -0.995$, $\lambda_{2,3} = 0.187 \pm 2.371i$. The type of the critical point is unstable saddle-focus.

There are three periodic solutions in Example 3. Periodic solutions are stable attractors. The solutions are depicted in Figure 34 and Figure 35.

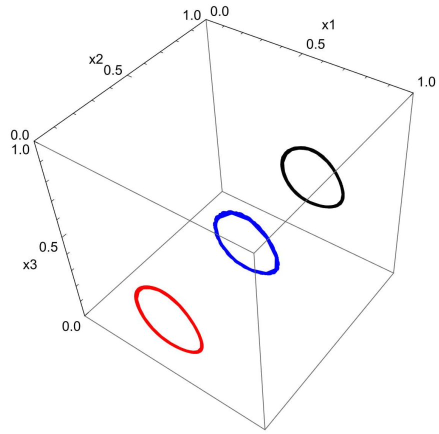

Figure 34: Example of two 3D limit cycles in the system (19) with the regulatory matrix (35).

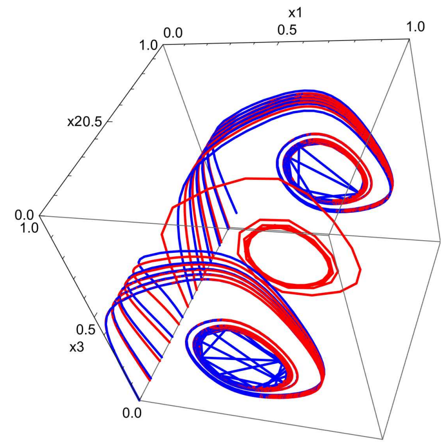

Figure 35: Three periodic solutions of the system (19) with the regulatory matrix (35).

Assume now that $w_{21} = 1$ in the matrix (35). Then there is only one critical point $(0.5367; 0.9999998; 0.3464)$. The standard linearization analysis provides the characteristic numbers $\lambda_1 = -0.999997$, $\lambda_{2,3} = 0.18762 \pm 2.37198i$. The type of critical point is unstable saddle-focus. The nullclines and one periodic solution are depicted consequently in Figure 36 and Figure 37.

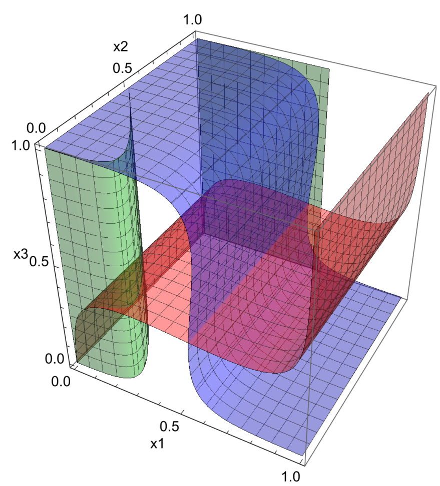

Figure 36: Nullclines x_1 - red, x_2 - green, x_3 - blue of the system (19) with the regulatory matrix (35), $w_{21} = 1$.

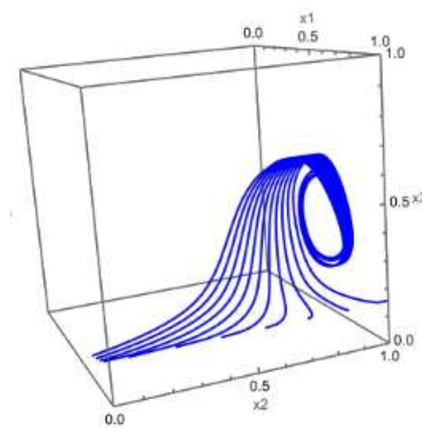

Figure 37: Some trajectories, tending to the periodic solution of the system (19) with the regulatory matrix (35), $w_{21} = 1$.

Assume that $w_{21} = 0.5$ and $w_{23} = -0.5$ then the middle periodic solution disappears. The nullclines and two periodic attractive solutions are depicted in Figure 38 and Figure 39.

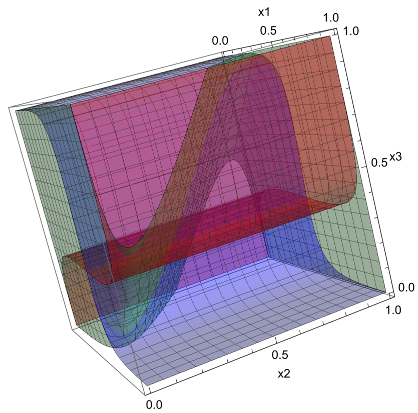

Figure 38: Nullclines x_1 - red, x_2 - green, x_3 - blue of the system (19) with the regulatory matrix (35),$w_{21} = 0.5$, $w_{23} = -0.5$.

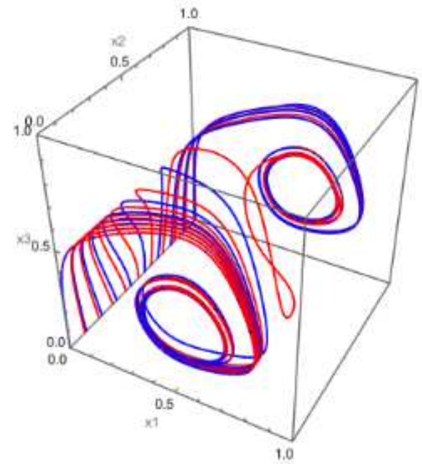

Figure 39: Some trajectories, tending to one of two periodic solutions of the system (19) with the regulatory matrix (35), $w_{21} = 0.5$, $w_{23} = -0.5$.

The dynamics of Lyapunov exponents are shown in Figure 40.

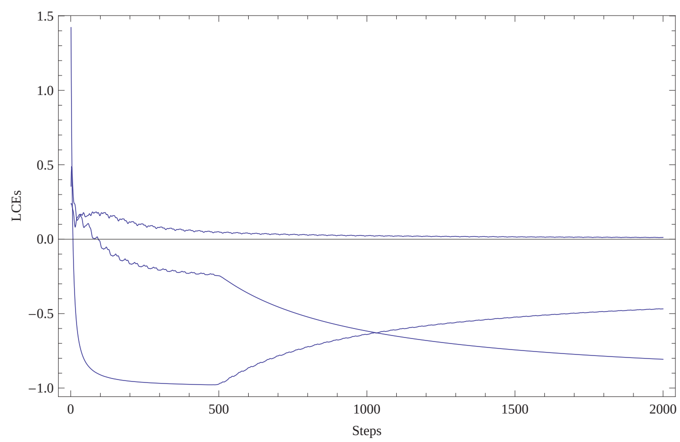

Figure 40: $LE_1 = 0.01, LE_2 = -0.47, LE_3 = -0.81$

5.8 Chaotic attractors

5.8.1 The quadratic jerk system

Consider

$$\begin{cases} \dfrac{dx_1}{dt} = x_2, \\ \dfrac{dx_2}{dt} = x_3, \\ \dfrac{dx_3}{dt} = -2x_1 - x_2 - 1.1x_3 - 0.3x_3^2 + x_1x_2. \end{cases} \tag{37}$$

The initial conditions are

$$x_1(0) = 0.1;\ x_2(0) = 0.1;\ x_3(0) = 0.1.$$

A chaotic attractor of the system (37) is shown in Figure 41.

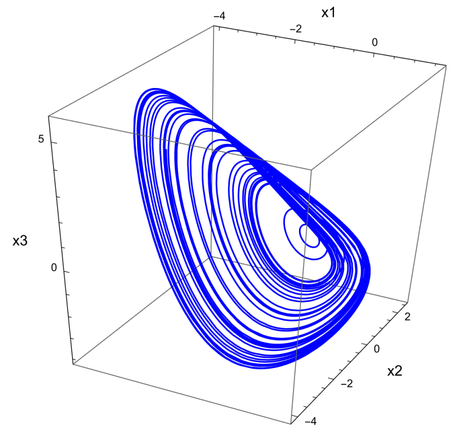

Figure 41: The self-excited chaotic attractor of the system (37).

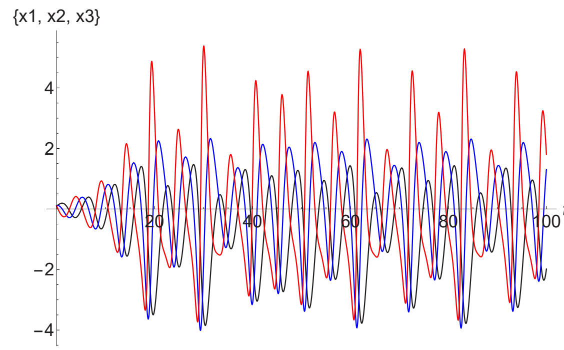

Figure 42: The graphs of $x_i(t), i = 1, 2, 3$, of the system (37).

The respective three-dimensional system was considered in the article [38].

The dynamics of Lyapunov exponents are shown in Figure 43.

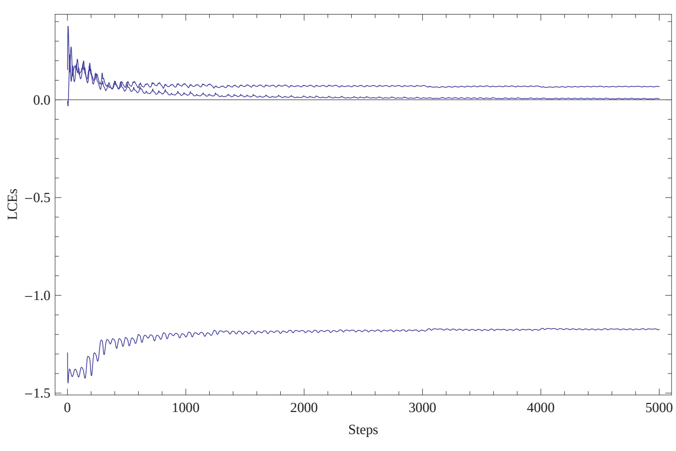

Figure 43: $LE_1 = 0.08, LE_2 = 0.00, LE_3 = -1.18$

There is one positive Lyapunov exponent that is why the self-excited attractor is chaotic. The system is a dissipative dynamical system, because the values of all Lyapunov exponents should sum to a negative number ($LE_1 + LE_2 + LE_3 = -1.1$).

5.8.2 The modified Das system

Consider

$$\mu_1 = \mu_2 = 7, \mu_3 = 13, v_1 = 0.65, v_2 = 0.42, v_3 = 0.1, \theta_1 = 0.5, \theta_2 = 0.3, \theta_3 = 0.7 \quad (38)$$

$$W = \begin{pmatrix} 0 & 1 & -5.65 \\ 1 & 0 & 0.135 \\ 1 & 0.02 & 0.03 \end{pmatrix}. \quad (39)$$

The initial conditions are

$$x_1(0) = 0.3;\ x_2(0) = 1.5;\ x_3(0) = 0.2. \quad (40)$$

The characteristic equation for critical point $(0.370457; 1.59272; 0.222436)$ is

$$-\lambda^3 + A\lambda^2 + B\lambda + C = 0,$$

where $A = -1.16152,\ B = -0.430187$ and $C = -0.688906$.

Solving the equation we have $\lambda_1 = -1.2558,\ \lambda_{2,3} = 0.0471391 \pm 0.739161i$. The type of critical point is an unstable saddle-focus. The system is chaotic in the sense that solutions exhibit non-regular behavior. The self-excited chaotic attractor is depicted in Figure 44.

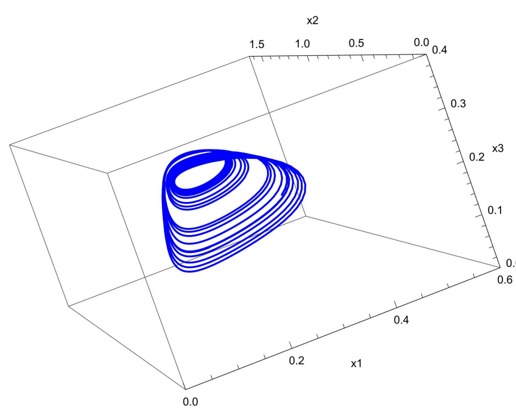

Figure 44: The self-excited chaotic attractor of the system (19) with the regulatory matrix (39).

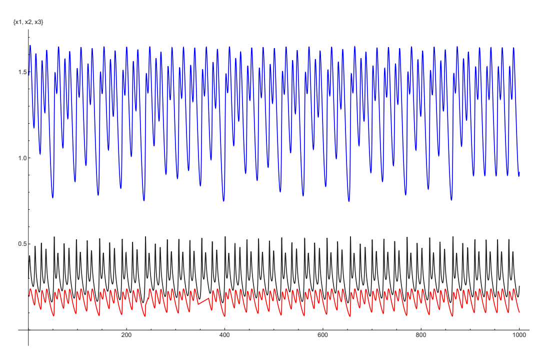

Figure 45: The graphs of $x_i(t), i = 1, 2, 3,$ of the system (19) with the regulatory matrix (39).

The respective three-dimensional system was studied in [13], [14].

Now we change the parameter w_{23} (that is, the third element in the second row) in the regulatory matrix (39). The coordinates of a single critical point, values of the characteristic numbers for this point, are provided. Computations are performed using Wolfram Mathematica.

Table 1. Results of calculations for the system (19) with regulatory matrix (39), changing the parameter w_{23}.

w_{23}	x^*	y^*	z^*	Real λ	Complex λ $\mathbb{R}$ part	Complex λ im part
0.0	0.3651	1.4571	0.1989	-1.4269	0.1322	0.6634
0.05	0.3671	1.5057	0.2073	-1.3714	0.1047	0.6886
0.10	0.3691	1.5562	0.2161	-1.3069	0.0726	0.71698
0.12	0.3699	1.57699	0.2197	-1.2783	0.0583	0.7294
0.13	0.3703	1.5875	0.2215	-1.2634	0.0519	0.7359
0.132	0.3703	1.5895	0.2219	-1.2604	0.0494	0.7371
0.133	0.3704	1.5906	0.2221	-1.2589	0.0487	0.7378
0.134	0.3704	1.5917	0.2223	-1.2573	0.0479	0.7385
0.136	0.3705	1.5938	0.2226	-1.2589	0.0487	0.7378
0.137	0.3705	1.5948	0.2228	-1.2527	0.0456	0.7405
0.138	0.3706	1.5959	0.22299	-1.2512	0.0448	0.7412
0.139	0.3706	1.5969	0.2232	-1.2494	0.0441	0.7418
0.14	0.3706	1.59799	0.2234	-1.2481	0.0433	0.7425
0.145	0.3708	1.6033	0.2243	-1.2403	0.0394	0.7459
0.15	0.3710	1.6087	0.2252	-1.2324	0.0354	0.7493
0.16	0.3714	1.6192	0.2270	-1.2162	0.0274	0.7564
0.18	0.3721	1.6406	0.2308	-1.1826	0.0107	0.7711
0.19	0.3725	1.6514	0.2326	-1.1652	0.002	0.7787
0.20	0.3729	1.6622	0.2345	-1.1473	-0.0069	0.7867

Table 2. Lyapunov exponents for the system (19) with regulatory matrix (39), 8000 steps

w_{23}	LE_1	LE_2	LE_3	$LE_1 + LE_2 + LE_3$
0.0	0.00228824	-0.133556	-1.03537	-1.16664
0.13	0.00174998	-0.0409256	-1.12505	-1.16423
0.132	0.00241997	-0.0284958	-1.13784	-1.16392
0.133	0.00405175	0.00091658	-1.16866	-1.1637
0.134	0.0200966	0.000487689	-1.18412	-1.16354
0.135	0.0162669	0.000848416	-1.18055	-1.16343
0.136	0.00335708	-0.0065914	-1.16009	-1.16332
0.137	-0.000688284	-0.0214113	-1.14116	-1.16326
0.19	-0.00174416	-0.0102177	-1.14928	-1.16124
0.20	-0.00816703	-0.0105543	-1.14236	-1.16108
1	-0.373617	-0.37358	-0.409939	-1.15714

Calculations showed the following:

- if $0 \leq w_{23} < 0.132$, then the system (39) has a periodic solution;
- if $0.133 < w_{23} \leq 0.135$, then the system (39) has a chaotic solution;
- if $0.136 < w_{23} \leq 0.19$, then the system (39) has a periodic solution;
- if $w_{23} > 0.2$, then the system (39) has a stable fixed point.

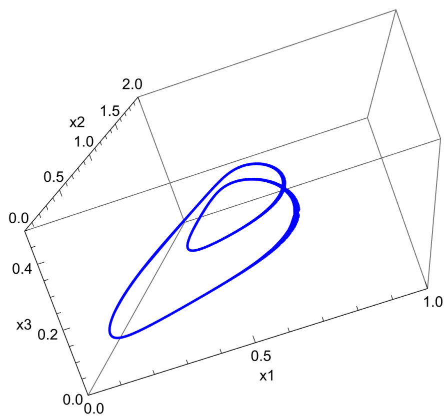

Figure 46: The periodic solution of the system (19) with the regulatory matrix (39), $w_{23} = 0.05$.

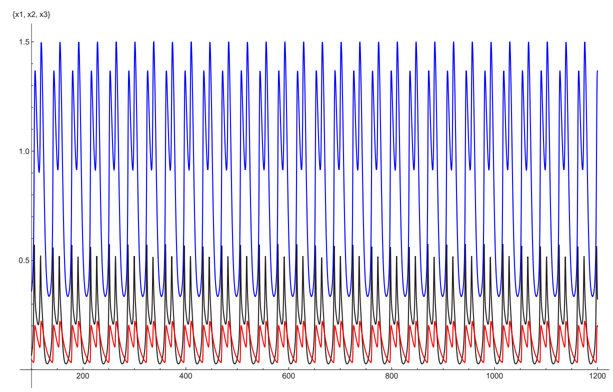

Figure 47: Solutions $(x_1(t), x_2(t), x_3(t))$ of the system (19) with the regulatory matrix (39), $w_{23} = 0.05$.

Now let us change w_{32} values in the regulatory matrix (39).

Table 3. Results of calculations for the system (19) with regulatory matrix (39), changing the parameter w_{32}.

w_{23}	x^*	y^*	z^*	Real λ	Complex λ $\mathbb{R}$ part	Complex λ im part
0.0	0.4092	1.7387	0.2449	-1.036	-0.0623	0.8666
0.01	0.3892	1.6656	0.2337	-1.1554	-0.0029	0.7966
0.03	0.3530	1.5213	0.2114	-1.3366	0.0873	0.6912
0.04	0.3368	1.4523	0.2007	-1.3996	0.1186	0.6507

Table 4. Lyapunov exponents for the system (19) with regulatory matrix (39), 8000 steps

w_{23}	LE_1	LE_2	LE_3	$LE_1 + LE_2 + LE_3$
0.0	-0.063377	-0.0643758	-0.406599	-1.16067
0.01	-0.00456357	-0.00807802	-1.14848	-1.16113
0.02	0.0162669	0.000848416	-1.18055	-1.16343
0.03	0.0015434	-0.186553	-0.980366	-1.16538
0.04	0.00381232	-0.0985729	-1.07168	-1.16644

Calculations showed the following:

- if $0 \leq w_{32} \leq 0.01$, then the system (39) has a stable fixed point;
- if $w_{32} = 0.02$, then the system (39) has a chaotic solution;
- if $0.03 \leq w_{32} \leq 0.04$, then the system (39) has a periodic solution.

From calculations we see that small changes in parameter values change the behavior of the system.

5.9 Conclusions for three-dimensional systems

The following are true for the system (19):

- the three-dimensional system (19) can have attractors of various kinds;
- the three-dimensional system (19) can have a single attracting point, multiple stable equilibria;
- the three-dimensional system (19) can have several stable periodic solutions, which serve as attractors;
- the self-excited chaotic attractor is possible.

6 Four-dimensional (4D) systems

Consider four-dimensional system

$$\begin{cases} \dfrac{dx_1}{dt} = \dfrac{1}{1+e^{-\mu_1(w_{11}x_1+w_{12}x_2+w_{13}x_3+w_{14}x_4-\theta_1)}} - v_1x_1, \\ \dfrac{dx_2}{dt} = \dfrac{1}{1+e^{-\mu_2(w_{21}x_1+w_{22}x_2+w_{23}x_3+w_{24}x_4-\theta_2)}} - v_2x_2, \\ \dfrac{dx_3}{dt} = \dfrac{1}{1+e^{-\mu_3(w_{31}x_1+w_{32}x_2+w_{33}x_3+w_{34}x_4-\theta_2)}} - v_3x_3, \\ \dfrac{dx_4}{dt} = \dfrac{1}{1+e^{-\mu_4(w_{41}x_1+w_{42}x_2+w_{43}x_3+w_{44}x_4-\theta_4)}} - v_4x_4. \end{cases} \tag{41}$$

The nullclines are given by

$$\begin{cases} v_1x_1 = \dfrac{1}{1+e^{-\mu_1(w_{11}x_1+w_{12}x_2+w_{13}x_3+w_{14}x_4-\theta_1)}}, \\ v_2x_2 = \dfrac{1}{1+e^{-\mu_2(w_{21}x_1+w_{22}x_2+w_{23}x_3+w_{24}x_4-\theta_2)}}, \\ v_3x_3 = \dfrac{1}{1+e^{-\mu_3(w_{31}x_1+w_{32}x_2+w_{33}x_3+w_{34}x_4-\theta_2)}}, \\ v_4x_4 = \dfrac{1}{1+e^{-\mu_4(w_{41}x_1+w_{42}x_2+w_{43}x_3+w_{44}x_4-\theta_4)}}. \end{cases} \tag{42}$$

Critical points are solutions of the system (42).

6.1 Linearized system

The linearized system for critical point $(x_1^*, x_2^*, x_3^*, x_4^*)$ is

$$\begin{cases} u_1' = -v_1u_1 + \mu_1w_{11}g_1u_1 + \mu_1w_{12}g_1u_2 + \mu_1w_{13}g_1u_3 + \mu_1w_{14}g_1u_4, \\ u_2' = -v_2u_2 + \mu_2w_{21}g_2u_1 + \mu_2w_{22}g_2u_2 + \mu_2w_{23}g_2u_3 + \mu_2w_{24}g_2u_4, \\ u_3' = -v_3u_3 + \mu_3w_{31}g_3u_1 + \mu_3w_{32}g_3u_2 + \mu_3w_{33}g_3u_3 + \mu_3w_{34}g_3u_4, \\ u_4' = -v_4u_4 + \mu_4w_{41}g_4u_1 + \mu_4w_{42}g_4u_2 + \mu_4w_{34}g_4u_3 + \mu_4w_{44}g_4u_4, \end{cases}$$

where

$$g_1 = \frac{e^{-\mu_1(w_{11}x_1^*+w_{12}x_2^*+w_{13}x_3^*+w_{14}x_4^*-\theta_1)}}{[1+e^{-\mu_1(w_{11}x_1^*+w_{12}x_2^*+w_{13}x_3^*+w_{14}x_4^*-\theta_1)}]^2},$$

$$g_2 = \frac{e^{-\mu_2(w_{21}x_1^*+w_{22}x_2^*+w_{23}x_3^*+w_{24}x_4^*-\theta_2)}}{[1+e^{-\mu_2(w_{21}x_1^*+w_{22}x_2^*+w_{23}x_3^*+w_{24}x_4^*-\theta_2)}]^2},$$
$$g_3 = \frac{e^{-\mu_3(w_{31}x_1^*+w_{32}x_2^*+w_{33}x_3^*+w_{34}x_4^*-\theta_3)}}{[1+e^{-\mu_3(w_{31}x_1^*+w_{32}x_2^*+w_{33}x_3^*+w_{34}x_4^*-\theta_3)}]^2},$$
$$g_4 = \frac{e^{-\mu_4(w_{41}x_1^*+w_{42}x_2^*+w_{43}x_3^*+w_{44}x_4^*-\theta_4)}}{[1+e^{-\mu_4(w_{41}x_1^*+w_{42}x_2^*+w_{43}x_3^*+w_{44}x_4^*-\theta_4)}]^2}.$$

The characteristic equation is

$$\lambda^4 + A\lambda^3 + B\lambda^2 + M\lambda + L = 0, \tag{43}$$

where

$$A = (v_1+v_2+v_3+v_4) - g_1w_{11}\mu_1 - g_2w_{22}\mu_2 - g_3w_{33}\mu_3 - g_4\mu_4w_{44},$$

$$\begin{aligned}
B = {} & v_3v_4 - g_1v_3w_{11}\mu_1 - g_1v_4w_{11}\mu_1 - g_2v_3w_{22}\mu_2 - g_2v_4w_{22}\mu_2 - g_1g_2w_{21}w_{12}\mu_1\mu_2 \\
& + g_1g_2w_{11}w_{22}\mu_1\mu_2 - g_3v_4w_{33}\mu_3 - g_1g_3w_{31}w_{13}\mu_1\mu_3 + g_1g_3w_{11}w_{33}\mu_1\mu_3 \\
& - g_2g_3w_{32}w_{23}\mu_2\mu_3 + g_2g_3w_{22}w_{33}\mu_2\mu_3 - g_1g_4w_{41}\mu_1\mu_4w_{14} - g_2g_4w_{42}\mu_2\mu_4w_{24} \\
& - g_3g_4w_{43}\mu_3\mu_4w_{34} - g_4v_3\mu_4w_{44} + g_1g_4w_{11}\mu_1\mu_4w_{44} + g_2g_4w_{22}\mu_2\mu_4w_{44} + g_3g_4w_{33}\mu_3\mu_4w_{44} \\
& + v_2(v_3+v_4-g_1w_{11}\mu_1-g_3w_{33}\mu_3-g_4\mu_4w_{44}) + v_1(v_2+v_3+v_4-g_2w_{22}\mu_2-g_3w_{33}\mu_3-g_4\mu_4w_{44}),
\end{aligned}$$

$$\begin{aligned}
M = {} & - g_1v_3v_4w_{11}\mu_1 - g_2v_3v_4w_{22}\mu_2 - g_1g_2v_3w_{21}w_{12}\mu_1\mu_2 - g_1g_2v_4w_{21}w_{12}\mu_1\mu_2 \\
& + g_1g_2v_3w_{11}w_{22}\mu_1\mu_2 + g_1g_2v_4w_{11}w_{22}\mu_1\mu_2 - g_1g_3v_4w_{31}w_{13}\mu_1\mu_3 \\
& + g_1g_3v_4w_{11}w_{33}\mu_1\mu_3 - g_2g_3v_4w_{32}w_{23}\mu_2\mu_3 + g_2g_3v_4w_{22}w_{33}\mu_2\mu_3 \\
& + g_1g_2g_3w_{31}w_{22}w_{13}\mu_1\mu_2\mu_3 - g_1g_2g_3w_{21}w_{32}w_{13}\mu_1\mu_2\mu_3 - g_1g_2g_3w_{31}w_{12}w_{23}\mu_1\mu_2\mu_3 \\
& + g_1g_2g_3w_{11}w_{32}w_{23}\mu_1\mu_2\mu_3 + g_1g_2g_3w_{21}w_{12}w_{33}\mu_1\mu_2\mu_3 - g_1g_2g_3w_{11}w_{22}w_{33}\mu_1\mu_2\mu_3 \\
& - g_1g_4v_3w_{41}\mu_1\mu_4w_{14} + g_1g_2g_4w_{41}w_{22}\mu_1\mu_2\mu_4w_{14} - g_1g_2g_4w_{21}w_{42}\mu_1\mu_2\mu_4w_{14} \\
& + g_1g_3g_4w_{41}w_{33}\mu_1\mu_3\mu_4w_{14} - g_1g_3g_4w_{31}w_{43}\mu_1\mu_3\mu_4w_{14} - g_2g_4v_3w_{42}\mu_2\mu_4w_{24} \\
& - g_1g_2g_4w_{41}w_{12}\mu_1\mu_2\mu_4w_{24} + g_1g_2g_4w_{11}w_{42}\mu_1\mu_2\mu_4w_{24} + g_2g_3g_4w_{42}w_{33}\mu_2\mu_3\mu_4w_{24} \\
& - g_2g_3g_4w_{32}w_{43}\mu_2\mu_3\mu_4w_{24} - g_1g_3g_4w_{41}w_{13}\mu_1\mu_3\mu_4w_{34} + g_1g_3g_4w_{11}w_{43}\mu_1\mu_3\mu_4w_{34} \\
& - g_2g_3g_4w_{42}w_{23}\mu_2\mu_3\mu_4w_{34} + g_2g_3g_4w_{22}w_{43}\mu_2\mu_3\mu_4w_{34} + g_1g_4v_3w_{11}\mu_1\mu_4w_{44} \\
& + g_2g_4v_3w_{22}\mu_2\mu_4w_{44} + g_1g_2g_4w_{21}w_{12}\mu_1\mu_2\mu_4w_{44} - g_1g_2g_4w_{11}w_{22}\mu_1\mu_2\mu_4w_{44} \\
& + g_1g_3g_4w_{31}w_{13}\mu_1\mu_3\mu_4w_{44} - g_1g_3g_4w_{11}w_{33}\mu_1\mu_3\mu_4w_{44} + g_2g_3g_4w_{32}w_{23}\mu_2\mu_3\mu_4w_{44} \\
& - g_2g_3g_4w_{22}w_{33}\mu_2\mu_3\mu_4w_{44} + v_1(v_3v_4 - g_2v_3w_{22}\mu_2 - g_2v_4w_{22}\mu_2 - g_3v_4w_{33}\mu_3 \\
& - g_2g_3w_{32}w_{23}\mu_2\mu_3 + g_2g_3w_{22}w_{33}\mu_2\mu_3 - g_2g_4w_{42}\mu_2\mu_4w_{24} \\
& - g_3g_4w_{43}\mu_3\mu_4w_{34} - g_4v_3\mu_4w_{44} + g_2g_4w_{22}\mu_2\mu_4w_{44} \\
& + g_3g_4w_{33}\mu_3\mu_4w_{44} + v_2(v_3+v_4-g_3w_{33}\mu_3-g_4\mu_4w_{44})) \\
& + v_2(v_3(v_4-g_1w_{11}\mu_1-g_4\mu_4w_{44}) - g_1\mu_1(v_4w_{11}+g_3w_{31}w_{13}\mu_3-g_3w_{11}w_{33}\mu_3+g_4w_{41}\mu_4w_{14} \\
& - g_4w_{11}\mu_4w_{44}) - g_3\mu_3(v_4w_{33}+g_4w_{43}\mu_4w_{34}-g_4w_{33}\mu_4w_{44})),
\end{aligned}$$

$$
\begin{aligned}
L = {} & v_1(v_2(v_3(v_4 - g_4\mu_4 w_{44}) - g_3\mu_3(v_4 w_{33} + g_4 w_{43}\mu_4 w_{34} - g_4 w_{33}\mu_4 w_{44})) \\
& - g_2\mu_2(v_3(v_4 w_{22} + g_4\mu_4(w_{42}w_{24} - w_{22}w_{44})) + g_3\mu_3(v_4(w_{32}w_{23} - w_{22}w_{33}) \\
& + g_4\mu_4(-w_{42}w_{33}w_{24} + w_{32}w_{43}w_{24} + w_{42}w_{23}w_{34} - w_{22}w_{43}w_{34} - w_{32}w_{23}w_{44} + w_{22}w_{33}w_{44})))) \\
& - g_1\mu_1(v_2(v_3(v_4 w_{11} + g_4\mu_4(w_{41}w_{14} - w_{11}w_{44})) + g_3\mu_3(v_4(w_{31}w_{13} - w_{11}w_{33}) \\
& + g_4\mu_4(-w_{41}w_{33}w_{14} + w_{31}w_{43}w_{14} + w_{41}w_{13}w_{34} - w_{11}w_{43}w_{34} - w_{31}w_{13}w_{44} + w_{11}w_{33}\ w_{44}))) \\
& + g_2\mu_2(v_3(v_4(w_{21}w_{12} - w_{11}w_{22}) \\
& + g_4\mu_4(-w_{41}w_{22}w_{14} + w_{21}w_{42}\ w_{14} + w_{41}w_{12}w_{24} - w_{11}w_{42}w_{24} - \ w_{21}w_{12}w_{44} + w_{11}w_{22}w_{44})) \\
& + g_3\mu_3(v_4(-w_{31}w_{22}w_{13} + w_{21}\ w_{32}w_{13} + w_{31}w_{12}w_{23} - w_{11}\ w_{32}w_{23} - w_{21}w_{12}w_{33} + w_{11}\ w_{22}w_{33}) \\
& + g_4\mu_4(-w_{21}w_{42}w_{33}w_{14} + w_{21}w_{32}w_{43}w_{14} + w_{11}w_{42}w_{33}w_{24} - w_{11}w_{32}w_{43}w_{24} \\
& + w_{21}w_{42}w_{13}w_{34} - w_{11}w_{42}w_{23}w_{34} - w_{21}w_{12}w_{43}w_{34} + w_{11}w_{22}w_{43}w_{34} \\
& + w_{41}(-w_{32}w_{23}w_{14} + w_{22}w_{33}w_{14} + w_{32}w_{13}w_{24} - w_{12}w_{33}w_{24} - w_{22}w_{13}w_{34} + w_{12}w_{23}w_{34}) \\
& - w_{21}w_{32}w_{13}w_{44} + w_{11}w_{32}w_{23}w_{44} + w_{21}\ w_{12}w_{33}w_{44} - w_{11}w_{22}w_{33}w_{44} \\
& + w_{31}(w_{42}w_{23}w_{14} - w_{22}w_{43}w_{14} - w_{42}w_{13}w_{24} + w_{12}w_{43}w_{24} + w_{22}w_{13}w_{44} - w_{12}w_{23}w_{44}))))).
\end{aligned}
$$

6.2 Critical points

The four-dimensional system has 4 eigenvalues.

- **4D node**. All eigenvalues are real and have the same sign. The node is stable (unstable) when the eigenvalues are negative (positive).
- **4D star**. All eigenvalues are equal. The 4D star is stable (unstable) when the eigenvalues are negative (positive).
- **Saddle**. All eigenvalues are real and at least one of them is positive and at least one is negative. Saddles are always unstable.
- **Focus – Node**. It has two real eigenvalues and a pair of complex-conjugate eigenvalues, and all eigenvalues have real parts of the same sign. The critical point is stable (unstable) when the sign is negative (positive).
- **Node – Focus**. It has two real negative eigenvalues and a pair of complex-conjugate eigenvalues with positive real part. The critical point is unstable.
- **Saddle – Focus**. Two real eigenvalues have different signs and complex-conjugate eigenvalues with positive or negative real part. The critical point is unstable.
- **Focus – Focus**. Two pairs of complex-conjugate eigenvalues. The critical point is stable when the signs of real parts are negative. The critical point is unstable when there is at least one positive real part.

6.3 Particular cases

6.3.1 Case 1

Let $w_{11} = w_{22} = w_{33} = w_{44} = 0$. The regulatory matrix is

$$W = \begin{pmatrix} 0 & w_{12} & w_{13} & w_{14} \\ w_{21} & 0 & w_{23} & w_{24} \\ w_{31} & w_{32} & 0 & w_{34} \\ w_{41} & w_{42} & w_{43} & 0 \end{pmatrix} \tag{44}$$

and the system of differential equations takes the form

$$\begin{cases} x_1' = \dfrac{1}{1 + e^{(w_{12}x_2 + w_{13}x_3 + w_{14}x_4 - \theta_1)}} - v_1 x_1, \\ x_2' = \dfrac{1}{1 + e^{-\mu_2(w_{21}x_1 + w_{23}x_3 + w_{24}x_4 - \theta_2)}} - v_2 x_2, \\ x_3' = \dfrac{1}{1 + e^{-\mu_3(w_{31}x_1 + w_{32}x_2 + w_{34}x_4 - \theta_2)}} - v_3 x_3, \\ x_4' = \dfrac{1}{1 + e^{-\mu_4(w_{41}x_1 + w_{42}x_2 + w_{43}x_3 - \theta_4)}} - v_4 x_4. \end{cases}$$

The linearized system for a critical point $(x_1^*, x_2^*, x_3^*, x_4^*)$ is then

$$\begin{cases} u_1' = -v_1 u_1 + \mu_1 w_{12} g_1 u_2 + \mu_1 w_{13} g_1 u_3 + \mu_1 w_{14} g_1 u_4, \\ u_2' = -v_2 u_2 + \mu_2 w_{21} g_2 u_1 + \mu_2 w_{23} g_2 u_3 + \mu_2 w_{24} g_2 u_4, \\ u_3' = -v_3 u_3 + \mu_3 w_{31} g_3 u_1 + \mu_3 w_{32} g_3 u_2 + \mu_3 w_{34} g_3 u_4, \\ u_4' = -v_4 u_4 + \mu_4 w_{41} g_4 u_1 + \mu_4 w_{42} g_4 u_2 + \mu_4 w_{34} g_4 u_3, \end{cases} \tag{45}$$

where

$$g_1 = \frac{e^{-\mu_1(w_{12}x_2^* + w_{13}x_3^* + w_{14}x_4^* - \theta_1)}}{[1 + e^{-\mu_1(w_{12}x_2^* + w_{13}x_3^* + w_{14}x_4^* - \theta_1)}]^2}, \tag{46}$$

$$g_2 = \frac{e^{-\mu_2(w_{21}x_1^* + w_{23}x_3^* + w_{24}x_4^* - \theta_2)}}{[1 + e^{-\mu_2(w_{21}x_1^* + w_{23}x_3^* + w_{24}x_4^* - \theta_2)}]^2}, \tag{47}$$

$$g_3 = \frac{e^{-\mu_3(w_{31}x_1^* + w_{32}x_2^* + w_{34}x_4^* - \theta_3)}}{[1 + e^{-\mu_3(w_{31}x_1^* + w_{32}x_2^* + w_{34}x_4^* - \theta_3)}]^2}, \tag{48}$$

$$g_4 = \frac{e^{-\mu_4(w_{41}x_1^* + w_{42}x_2^* + w_{43}x_3^* - \theta_4)}}{[1 + e^{-\mu_4(w_{41}x_1^* + w_{42}x_2^* + w_{43}x_3^* - \theta_4)}]^2}. \tag{49}$$

The characteristic equation is (43), where

$$A = v_1 + v_2 + v_3 + v_4,$$

$$B = v_3 v_4 + v_2(v_3 + v_4) + v_1(v_2 + v_3 + v_4) - g_1 g_2 w_{21} w_{12} \mu_1 \mu_2 - g_1 g_3 w_{31} w_{13} \mu_1 \mu_3$$
$$- g_2 g_3 w_{32} w_{23} \mu_2 \mu_3 - g_1 g_4 w_{41} \mu_1 \mu_4 w_{14} - g_2 g_4 w_{42} \mu_2 \mu_4 w_{24} - g_3 g_4 w_{43} \mu_3 \mu_4 w_{34},$$

$$
\begin{aligned}
M =& - g_1g_2v_3w_{21}w_{12}\mu_1\mu_2 - g_1g_2v_4w_{21}w_{12}\mu_1\mu_2 - g_1g_3v_4w_{31}w_{13}\mu_1\mu_3 - g_2g_3v_4w_{32}w_{23}\mu_2\mu_3 \\
& -g_1g_2g_3w_{21}w_{32}w_{13}\mu_1\mu_2\mu_3 - g_1g_2g_3w_{31}w_{12}w_{23}\mu_1\mu_2\mu_3 - g_1g_4v_3w_{41}\mu_1\mu_4w_{14} \\
& -g_1g_2g_4w_{21}w_{42}\mu_1\mu_2\mu_4w_{14}g_1g_3g_4w_{31}w_{43}\mu_1\mu_3\mu_4w_{14} - g_2g_4v_3w_{42}\mu_2\mu_4w_{24} \\
& - g_1g_2g_4w_{41}w_{12}\mu_1\mu_2\mu_4w_{24} - g_2g_3g_4w_{32}w_{43}\mu_2\mu_3\mu_4w_{24} - g_1g_3g_4w_{41}w_{13}\mu_1\mu_3\mu_4w_{34} \\
& -g_2g_3g_4w_{42}w_{23}\mu_2\mu_3\mu_4w_{34} + v_2(v_3v_4 - g_1\mu_1(g_3w_{31}\mu_3w_{13} + g_4w_{41}\mu_4w_{14}) - g_3g_4w_{43}\mu_3\mu_4w_{34}) \\
& +v_1(v_3v_4 + v_2(v_3 + v_4) - g_2g_3w_{32}w_{23}\mu_2\mu_3 - g_2g_4w_{42}\mu_2\mu_4w_{24} - g_3g_4w_{43}\mu_3\mu_4w_{34}), \\
L =& v_1(v_2(v_3(v_4 - g_3g_4\mu_3\mu_4w_{34}) - g_2\mu_2(g_4v_3w_{42}w_{24}\mu_4(v_4w_{32}w_{23} + g_4\mu_4(w_{32}w_{43}w_{24} + w_{42}w_{23}w_{34})))) \\
& -g_1\mu_1(v_2(g_4v_3\mu_4(w_{41}w_{14} + g_3\mu_3(v_4(w_{31}w_{13} + g_4\mu_4(w_{31}w_{43}w_{14} + w_{41}w_{13}w_{34}))) \\
& +g_2\mu_2(v_3(v_4w_{21}w_{12} + g_4\mu_4(w_{31}w_{42}w_{14} + w_{41}w_{12}w_{24})) + g_3\mu_3(v_4(w_{31}w_{32}w_{13} + w_{31}w_{12}w_{23}) \\
& +g_4\mu_4(w_{31}w_{32}w_{43}w_{14} + w_{31}(w_{42}w_{23}w_{14} - w_{42}w_{13}w_{24} + w_{12}w_{43}w_{24}) + w_{21}w_{42}w_{13}w_{34} \\
& +w_{41}(-w_{32}w_{23}w_{14} + w_{12}w_{13}w_{24} + w_{12}w_{23}w_{34}))))).
\end{aligned}
$$

Example 1. Consider $\mu_1 = \mu_2 = \mu_3 = 8$, $\mu_4 = 15$, $v_1 = 0.1$, $v_2 = 0.3$, $v_3 = 1$, $v_4 = 0.4$ and $\theta_1 = 1.2, \theta_2 = -0.7, \theta_3 = 1.8, \theta_4 = -0.2$.

$$
W = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 0.3 & 0 & 1 & 0.4 \\ 0 & 1 & 0 & 1 \\ 0.2 & 1 & -1 & 0 \end{pmatrix}. \tag{50}
$$

The characteristic equation for critical point $(0.5053; 3.3333; 0.9999; 2.5)$ is (43), where $A = 1.8$, $B = 0.99$, $M = 0.202$ and $L = 0.012$.

Solving the equation we have $\lambda_1 = -0.1$, $\lambda_2 = -0.3$, $\lambda_2 = -0.4$ and $\lambda_4 = -1$. The type of the critical point is a 4D stable node.

6.3.2 Case 2

Let $v_1 = v_2 = v_3 = v_4 = 1$ and the regulatory matrix is (44).

The system of differential equations takes the form

$$
\begin{cases}
x_1' = \dfrac{1}{1 + e^{(w_{12}x_2 + w_{13}x_3 + w_{14}x_4 - \theta_1)}} - x_1, \\
x_2' = \dfrac{1}{1 + e^{-\mu_2(w_{21}x_1 + w_{23}x_3 + w_{24}x_4 - \theta_2)}} - x_2, \\
x_3' = \dfrac{1}{1 + e^{-\mu_3(w_{31}x_1 + w_{32}x_2 + w_{34}x_4 - \theta_2)}} - x_3, \\
x_4' = \dfrac{1}{1 + e^{-\mu_4(w_{41}x_1 + w_{42}x_2 + w_{43}x_3 - \theta_4)}} - x_4.
\end{cases}
$$

The linearized system for a critical point $(x_1^*, x_2^*, x_3^*, x_4^*)$ is then

$$
\begin{cases}
u_1' = -u_1 + \mu_1w_{12}g_1u_2 + \mu_1w_{13}g_1u_3 + \mu_1w_{14}g_1u_4, \\
u_2' = -u_2 + \mu_2w_{21}g_2u_1 + \mu_2w_{23}g_2u_3 + \mu_2w_{24}g_2u_4, \\
u_3' = -u_3 + \mu_3w_{31}g_3u_1 + \mu_3w_{32}g_3u_2 + \mu_3w_{34}g_3u_4, \\
u_4' = -u_4 + \mu_4w_{41}g_4u_1 + \mu_4w_{42}g_4u_2 + \mu_4w_{34}g_4u_3
\end{cases}
$$

and g_1, g_2, g_3, given in (46) to (49).

The characteristic equation is (43), where

$$A = 4,$$

$$\begin{aligned}B =& 6 - g_1g_2w_{21}w_{12}\mu_1\mu_2 - g_1g_3w_{31}w_{13}\mu_1\mu_3 - g_2g_3w_{32}w_{23}\mu_2\mu_3 - g_1g_4w_{41}\mu_1\mu_4w_{14}\\ &-g_2g_4w_{42}\mu_2\mu_4w_{24}g_3g_4w_{43}\mu_3\mu_4w_{34},\\ M =& 4 - 2g_1g_2w_{12}w_{21}\mu_1\mu_2 - g_1g_3w_{13}w_{31}\mu_1\mu_3 - 2g_2g_3w_{23}w_{32}\mu_2\mu_3 - g_1g_2g_3w_{12}w_{23}w_{31}\mu_1\mu_2\mu_3\\ &-g_1g_2g_3w_{13}w_{21}w_{32}\mu_1\mu_2\mu_3 - g_1g_4w_{41}\mu_1\mu_4w_{14} - 2g_2g_4w_{24}\mu_2\mu_4w_{42} - g_1g_2g_4w_{41}w_{12}\mu_1\mu_2\mu_4w_{24}\\ &-g_1g_2g_4w_{14}w_{21}\mu_1\mu_2\mu_4w_{424} - 2g_3g_4w_{34}w_{43}\mu_3\mu_4 - g_1g_3g_4w_{13}w_{34}\mu_1\mu_3\mu_4w_{41}\\ &-g_1g_3g_4w_{14}w_{31}\mu_1\mu_3\mu_4w_{43} - g_2g_3g_4w_{23}w_{34}\mu_2\mu_3\mu_4w_{42} - g_2g_3g_4w_{24}w_{32}\mu_2\mu_3\mu_4w_{43}\\ &-g_1\mu_1(g_3w_{13}w_{31}\mu_3 + g_4w_{14}w_{41}\mu_4),\\ L =& 1 - g_3g_4w_{34}w_{43}\mu_3\mu_4 - g_2\mu_2(g_4w_{24}w_{42}\mu_4 + g_3\mu_3(w_{23}w_{32} + g_4(w_{23}w_{34}w_{42} + w_{24}w_{32}w_{43}\mu_4))\\ &-g_1\mu_1(g_4w_{14}w_{41}\mu_4 + g_3\mu_3(w_{13}w_{31} + g_4(w_{13}w_{34}w_{41} + w_{14}w_{31}w_{43}\mu_4)\\ &+g_2\mu_2(w_{12}w_{21} + g_4(w_{12}w_{24}w_{41} + w_{14}w_{21}w_{42})\mu_4 + g_3\mu_3(w_{12}w_{23}w_{31} + w_{13}w_{21}w_{32}\\ &+g_4((-w_{14}w_{23}w_{32} + w_{13}w_{24}w_{32} + w_{12}w_{23}w_{34})w_{41} + w_{13}w_{21}w_{34}w_{42} + w_{14}w_{21}w_{32}w_{43}\\ &-w_{12}w_{21}w_{34}w_{43} + w_{31}(w_{14}w_{23}w_{42} - w_{13}w_{24}w_{42} + w_{12}w_{24}w_{43}))\mu_4))).\end{aligned}$$

Example 2. Consider $\mu_1 = \mu_2 = \mu_3 = 8$, $\mu_4 = 15$, $v_1 = v_2 = v_3 = v_4 = 1$, $\theta_1 = 1.2, \theta_2 = -0.7, \theta_3 = 1.8, \theta_4 = -0.2$ and the regulatory matrix is (50).

The characteristic equation for critical point $(0.0000698; 0.9999; 0.8278; 0.9963)$ is (43), where $A = 4$, $B = 6.06384$, $M = 4.12768$ and $L = 1.06384$.

Solving the equation we have $\lambda_1 = -1.00003$, $\lambda_2 = -0.999969$ and $\lambda_{2,3} = -1 \pm 0.2527i$. The type of the critical point is a stable focus-node.

6.3.3 Case 3

Let $v_1 = v_2 = v_3 = v_4 = 1$ and the regulatory matrix is

$$W = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{51}$$

and the system of differential equations takes the form

$$\begin{cases} x_1' = \dfrac{1}{1 + e^{-\mu_1(w_{11}x_1 - \theta_1)}} - x_1, \\ x_2' = \dfrac{1}{1 + e^{-\mu_2(w_{22}x_2 - \theta_2)}} - x_2, \\ x_3' = \dfrac{1}{1 + e^{-\mu_3(w_{33}x_3 - \theta_2)}} - x_3, \\ x_4' = \dfrac{1}{1 + e^{-\mu_4(w_{44}x_4 - \theta_4)}} - x_4. \end{cases}$$

The linearized system for critical point $(x_1^*, x_2^*, x_3^*, x_4^*)$ is

$$\begin{cases} u_1' = -u_1 + \mu_1 w_{11} g_1 u_1, \\ u_2' = -u_2 + \mu_2 w_{22} g_2 u_2, \\ u_3' = -u_3 + \mu_3 w_{33} g_3 u_3, \\ u_4' = -u_4 + \mu_4 w_{44} g_4 u_4, \end{cases}$$

where

$$g_1 = \frac{e^{-\mu_1(w_{11}x_1^* - \theta_1)}}{[1 + e^{-\mu_1(w_{11}x_1^* - \theta_1)}]^2},$$

$$g_2 = \frac{e^{-\mu_2(w_{22}x_2^* - \theta_2)}}{[1 + e^{-\mu_2(w_{22}x_2^* - \theta_2)}]^2},$$

$$g_3 = \frac{e^{-\mu_3(w_{33}x_3^* - \theta_3)}}{[1 + e^{-\mu_3(w_{33}x_3^* - \theta_3)}]^2},$$

$$g_4 = \frac{e^{-\mu_4(w_{44}x_4^* - \theta_4)}}{[1 + e^{-\mu_4(w_{44}x_4^* - \theta_4)}]^2}.$$

The characteristic equation is (43), where

$$A = 4 - g_1\mu_1 - g_2\mu_2 - g_3\mu_3 - g_4\mu_4,$$

$$\begin{aligned} B = {} & 6 - 3g_1\mu_1 - 3g_2\mu_2 + g_1g_2\mu_1\mu_2 - 3g_3\mu_3 + g_1g_3\mu_1\mu_3 + g_2g_3\mu_2\mu_3 - 3g_4\mu_4 \\ & + g_1g_4\mu_1\mu_4 + g_2g_4\mu_2\mu_4 + g_3g_4\mu_3\mu_4, \end{aligned}$$

$$\begin{aligned} M = {} & 4 - 2g_1\mu_1 - 3g_2\mu_2 + 2g_1g_2\mu_1\mu_2 - 2g_3\mu_3 + g_1g_3\mu_1\mu_3 + 2g_2g_3\mu_2\mu_3 - g_1g_2g_3\mu_1\mu_2\mu_3 \\ & - 3g_4\mu_4 + g_1g_4\mu_1\mu_4 + 2g_2g_4\mu_2\mu_4 - g_1g_2g_4\mu_1\mu_2\mu_4 + g_3g_4\mu_3\mu_4 - g_1g_3g_4\mu_1\mu_3\mu_4 \\ & - g_2g_3g_4\mu_2\mu_3\mu_4 - g_3\mu_3(1 - g_4\mu_4) - g_1\mu_1(1 - g_3\mu_3 - g_4\mu_4), \end{aligned}$$

$$\begin{aligned} L = {} & 1 - g_4\mu_4 - g_3\mu_3(1 - g_4\mu_4) - g_2\mu_2(1 - g_4\mu_4 + g_3\mu_3(-1 + g_4\mu_4)) \\ & - g_1\mu_1(1 - g_4\mu_4 + g_3\mu_3(-1 + g_4\mu_4) + g_2\mu_2(-1 + g_4\mu_4 + g_3\mu_3(1 - g_4\mu_4))). \end{aligned}$$

Example 3. Consider $\mu_1 = \mu_2 = \mu_3 = 8,\ \mu_4 = 15,\ v_1 = v_2 = v_3 = v_4 = 1,\ \theta_1 = 1.2, \theta_2 = -0.7, \theta_3 = 1.8, \theta_4 = -0.2$ and the regulatory matrix is (51).

The characteristic equation for a critical point $(0.0000678; 0.999999; 5.57393 \cdot 10^{-7}; 0.0.99999)$ is (43), where $A = 3.99944$, $B = 5.99833$, $M = 3.99833$ and $L = 0.999443$.

Solving the equation we have $\lambda_{1,2,3,4} = -0.999861$. The type of the critical point is a stable 4D star.

6.4 Lyapunov exponents

Relationships between the Lyapunov exponents and the properties and types of attractors:

- $(LE_1, LE_2, LE_3, LE_4) = (-,-,-,-)$ - stable fixed point;
- $(LE_1, LE_2, LE_3, LE_4) = (0,-,-,-)$ - periodic solutions (limit cycles);
- $(LE_1, LE_2, LE_3, LE_4) = (0,0,-,-)$ - quasiperiodic solution;
- $(LE_1, LE_2, LE_3, LE_4) = (+,0,-,-)$ - strange attractor;
- $(LE_1, LE_2, LE_3, LE_4) = (+,+,0,-)$ - hyperchaotic attractor [40].

6.5 Artificial Neural Networks

An artificial neural network (ANN) is a computational architecture for processing complex data using multiple interconnected processors and computational paths. Artificial neural networks, created by analogy with the human brain, can train and analyze large and complex data sets that are extremely difficult to process using more linear algorithms.

In 1943 American neurophysiologist and cybernetician Warren Sturgis McCulloch and American logician Walter Harry Pitts modeled a neuron as a switch that receives input from other neurons and, depending on the total weighted input, is either activated or remains inactive [31],[22].

In 1958 ANN was created by psychologist Frank Rosenblatt. It was called Perceptron and was designed to simulate the activity of the human brain in processing visual data and in learning to recognize objects. Subsequently, similar artificial neural networks were developed to study the process of cognition Over time, it became clear that in addition to analyzing the activity of the human brain, it can perform other very useful functions. Because of their ability to pattern-match and learn, these networks have been used to analyze many problems that are extremely difficult or impossible to solve using traditional computational or statistical methods [18].

In the late 80s, artificial neural networks began to be actively used for a variety of purposes [22]. For example, are used in the economy to increase the productivity of discount calculation, targeted marketing, and credit evaluation.

The principle of operation of an artificial neural network is to form connections between many different processing elements, each of which serves as an analog of one neuron in the brain of a biological being. Neurons can be physically reproduced or simulated using a digital computer. Each neuron receives a set of input signals, and then, taking into account the internal system of weight coefficients, generates one output signal, which, as a rule, serves as input for another neuron. Neurons are closely interconnected with each other and are organized into several different levels. The input layer receives the input data, and the output layer generates the final result. Typically, there are one or more hidden levels between these two levels. In such a structure, it is impossible to predict or know exactly how data is transmitted.

Definition 6.1. *An Artificial Neural Network is a mathematical model that tries to simulate the structure and functionalities of biological neural networks. A basic building block of every artificial neural network is an artificial neuron, that is, a simple mathematical model (function) [83].*

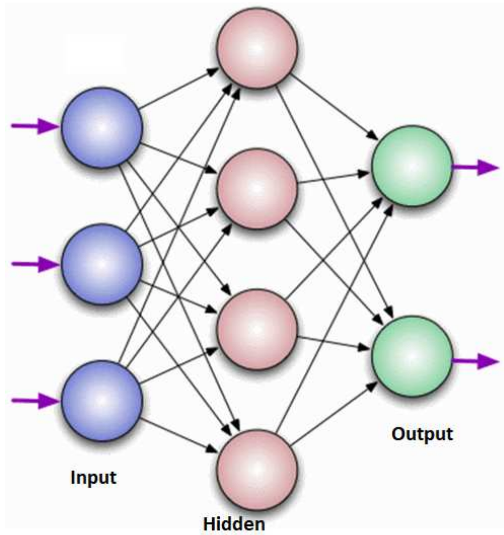

Figure 48: The example of ANN construction [37].

Neural networks consist of neurons interconnected, so the neuron is the main part of the neural network. The neurons only do two things: multiply the inputs by the weights and sum them up and add the bias, and the second action is the activation.

The input data is the data that the neuron receives from previous neurons or the user. Weights are assigned to each input of the neuron, initially, they are assigned random numbers. When training a neural network, the value of neurons and biases changes. The weights are multiplied by the input data that is fed to the neuron. The biases are assigned to each neuron, just like the initial bias weights, these are random numbers. Bias makes it easier and faster to train a neural network [18]. Typically this transformation involves the use of a sigmoid, hyperbolic-tangent, or other nonlinear function [93].

One example of which is

$$x_i' = \tanh \sum_{j=1}^{N} a_j x_j - b_i x_i, \tag{52}$$

where N is the number of neurons, each of which represents a dimension of the system [81]. The hyperbolic tangent is a sigmoid function.

Consider the system

$$\begin{cases} x_1' = \tanh(x_1 + x_2 + x_3 + x_4) - bx_1, \\ x_2' = \tanh(x_1 + x_2 + x_3 + x_4) - bx_2, \\ x_3' = \tanh(x_1 + x_2 + x_3 + x_4) - bx_3, \\ x_4' = \tanh(x_1 + x_2 + x_3 + x_4) - bx_4 \end{cases} \tag{53}$$

with the regulatory matrix

$$W = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix} \tag{54}$$

and $b = 0.03$.

The initial conditions are

$$x_1(0) = 1.2;\ x_2(0) = 0.4;\ x_3(0) = 1.2;\ x_4(0) = -1.$$

The graph of the regulatory matrix (54) is presented in Figure 49.

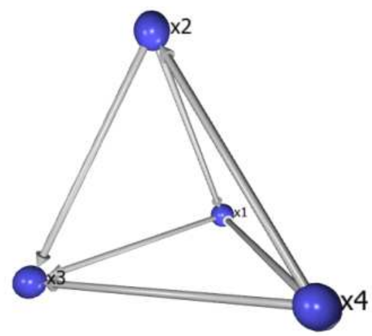

Figure 49: The graph, corresponding to the system (53), with the regulatory matrix (54).

The attractor is shown in Figure 50 and Figure 51 and the solutions in Figure 52 and Figure 53.

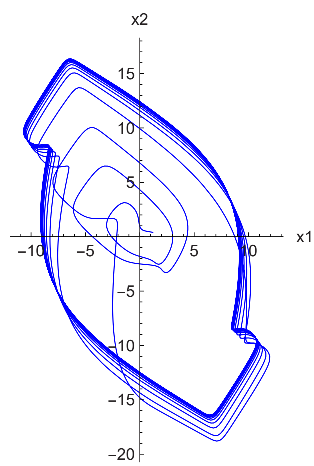

Figure 50: The projection of the attractor on 2D subspace $(x_1(t), x_2(t))$

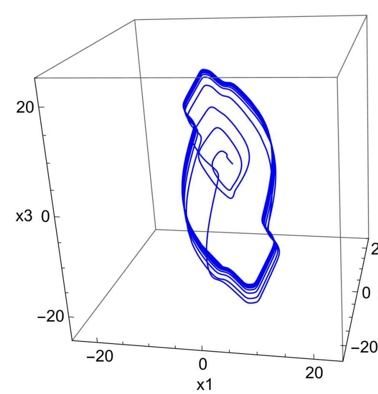

Figure 51: The projection of the attractor on 3D subspace $(x_1(t), x_2(t), x_3(t))$

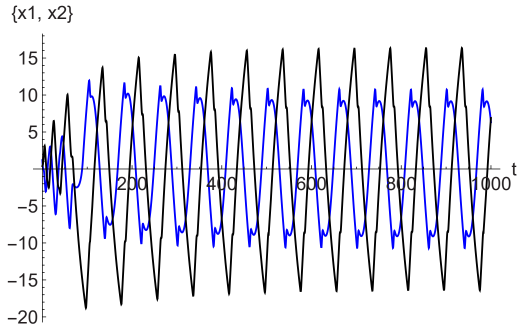

Figure 52: Solutions $(x_1(t), x_2(t))$ of the system (53) with the regulatory matrix (54).

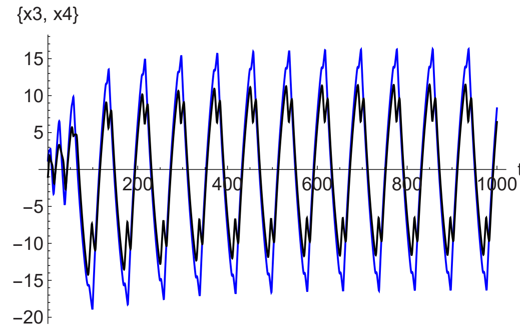

Figure 53: Solutions $(x_3(t), x_4(t))$ of the system (53) with the regulatory matrix (54).

For specific parameters solution of the system has a chaotic trajectory as shown in Figure 54 and Figure 55. The minimal dissipative artificial neural network that exhibits chaos has $N = 4$ and is given by system (53) with the regulatory matrix (54) and $b = 0.043$ and an attractor as shown in Figure 56 and Figure 57 [81].

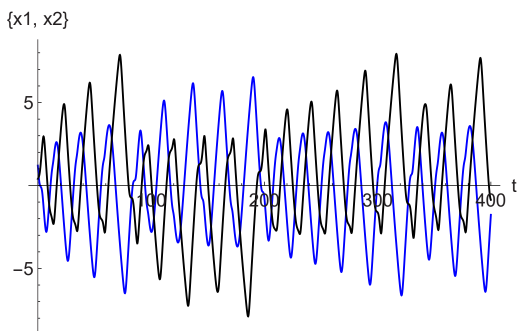

Figure 54: Solutions $(x_1(t), x_2(t))$ of the system (53) with the regulatory matrix (54), $b = 0.043$.

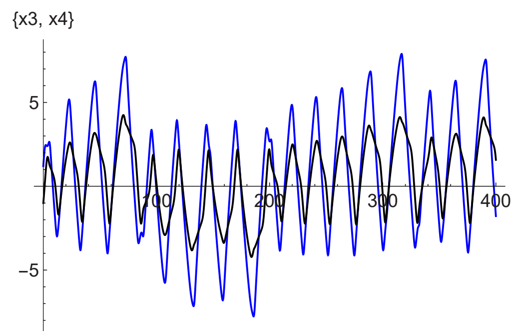

Figure 55: Solutions $(x_3(t), x_4(t))$ of the system (53) with the regulatory matrix (54), $b = 0.043$.

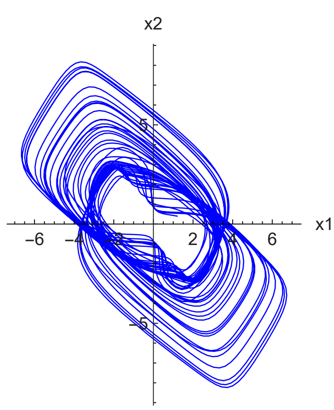

Figure 56: The projection of the attractor on 2D subspace $(x_1(t), x_2(t))$

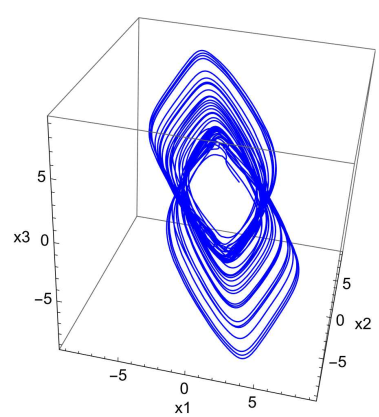

Figure 57: The projection of the attractor on 3D subspace $(x_1(t), x_2(t), x_3(t))$

The dynamics of Lyapunov exponents are shown in Figure 58.

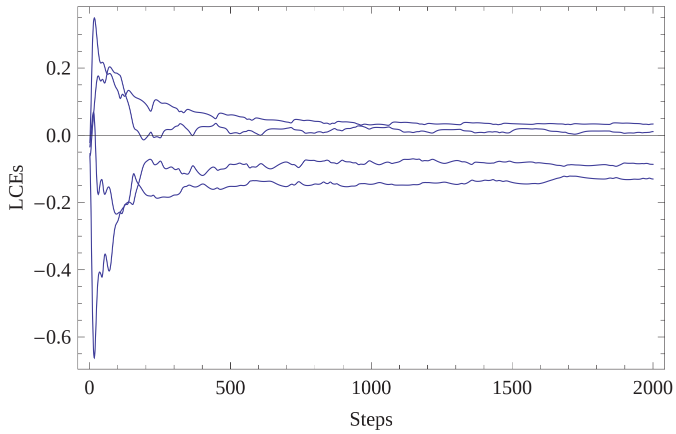

Figure 58: $LE_1 = 0.03, LE_2 = 0.01, LE_3 = -0.09, LE_4 = -0.13$

6.6 4D system from 2D and 2D systems

Let us return to the GRN system (41).

Example 1. Consider

$$W = \begin{pmatrix} k_1 & 2 & 0 & 0 \\ -2 & k_1 & 0 & 0 \\ 0 & 0 & k_2 & 2 \\ 0 & 0 & -2 & k_2 \end{pmatrix}, \quad (55)$$

where k_1 and k_2 are positive numbers.

The system then is uncoupled, consisting of two independent 2D systems. Each of these 2D systems is known to have a stable periodic trajectory, which surrounds an unstable 2D focus. The 4D system has the self-excited attractor, which is the product of two stable 2D limit cycles.

Example 2. Consider the system (41) with the regulatory matrix (55), where $k_1 = 0.5$ and $k_2 = 1.815$ and $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 10$, $v_1 = v_2 = v_3 = v_4 = 1$, $\theta_1 = 1.2$, $\theta_2 = -0.7$, $\theta_3 = 1.8$, $\theta_4 = -0.28$.

The initial conditions are

$$x_1(0) = 0.5;\ x_2(0) = 0.32;\ x_3(0) = 0.4;\ x_4(0) = 0.39.$$

This system consists of two independent two-dimensional systems. There is exactly one critical point. The standard linearization analysis provides the characteristic numbers $\lambda_{1,2} = 0.2469 \pm 4.9875i$; $\lambda_{3,4} = 3.4667 \pm 4.9215i$. The type of critical point is an unstable focus-focus.

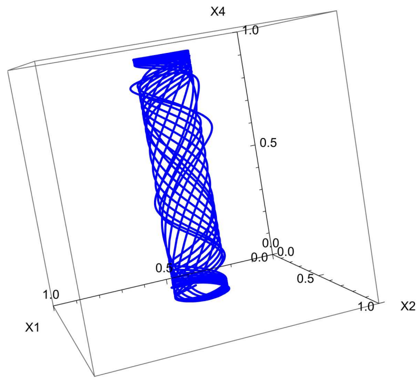

Figure 59: The projection of 4D trajectories to 3D subspace $(x_1(t), x_2(t), x_4(t))$.

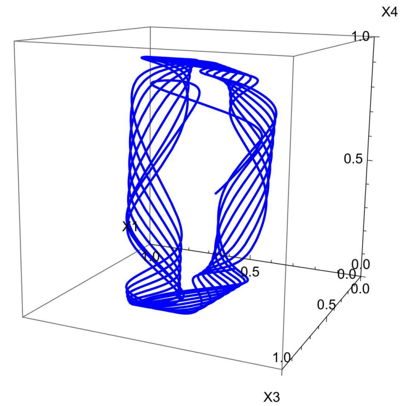

Figure 60: The projection of 4D trajectories to 3D subspace $(x_1(t), x_3(t), x_4(t))$.

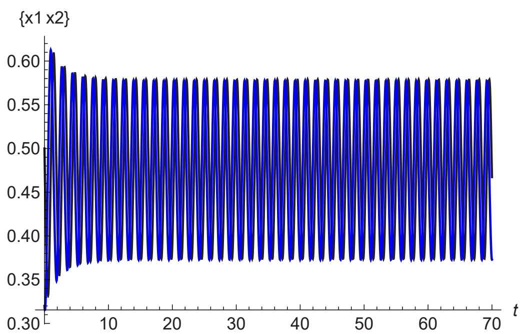

Figure 61: The graphs of periodic solutions $(x_1(t), x_2(t))$ of the system (41) with the regulatory matrix (55), $k_1 = 0.5$ and $k_2 = 1.815$.

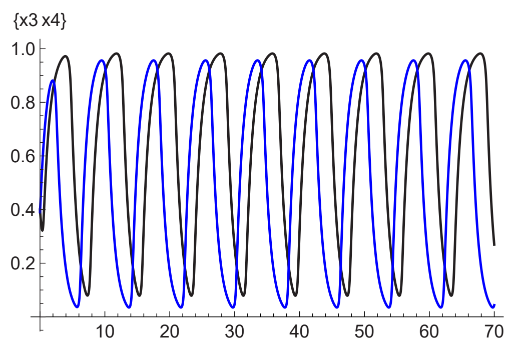

Figure 62: The graphs of periodic solutions $(x_3(t), x_4(t))$ of the system (41) with the regulatory matrix (55), $k_1 = 0.5$ and $k_2 = 1.815$.

Let us change two elements at the right upper (w_{14}) and left lower (w_{41}) corners. Let $w_{41} = 0.1$ and (w_{14}) values are considered in Table 5.

Table 5. Results of calculations for the system (41) with regulatory matrix (55) $k_1 = 0.5$ and $k_2 = 1.815$, changing the parameter w_{14}.

w_{14}	$\lambda_{1,2}$	$\lambda_{3,4}$	Lyapunov exponents
-1.2	$0.189 \pm 4.49i$	$3.374 \pm 4.912i$	(0; -0.48; -0.89; -0.96)
-1.1	$0.206 \pm 4.586i$	$3.379 \pm 4.908i$	(0; -0.70; -0.70; -0.87)
-1	$0.220 \pm 4.671i$	$3.384 \pm 4.905i$	(0.05; 0; -0.88; -0.98)
-0.9	$0.232 \pm 4.745i$	$3.389 \pm 4.902i$	(0; -0.27; -0.29; -0.89)
-0.8	$0.242 \pm 4.808i$	$3.394 \pm 4.899i$	(0; -0.05; -0.58; -0.88)
-0.7	$0.250 \pm 4.862i$	$3.399 \pm 4.897i$	(0.03; 0; -0.26; -0.89)
-0.6	$0.256 \pm 4.906i$	$3.405 \pm 4.896i$	(0; -0.20; -0.20; -0.89)
-0.5	$0.260 \pm 4.941i$	$3.410 \pm 4.894i$	(0; -0.09; -0.35; -0.89)
-0.4	$0.261 \pm 4.968i$	$3.415 \pm 4.893i$	(0; -0.13; -0.33; -0.89)

Calculations showed the following:

- if $-1.2 \leq w_{14} < -1$, then the system (41) with the regulatory matrix (55) has a periodic solution;
- if $w_{14} = -1$, then the system (41) with the regulatory matrix (55) is chaotic;
- if $-0.9 \leq w_{14} < -0.7$, then the system (41) with the regulatory matrix (55) has a periodic solution;
- if $w_{14} = -0.7$, then the system (41) with the regulatory matrix (55) is chaotic;
- if $-0.6 \leq w_{14} \leq -0.4$, then the system (41) with the regulatory matrix (55) has a periodic solution.

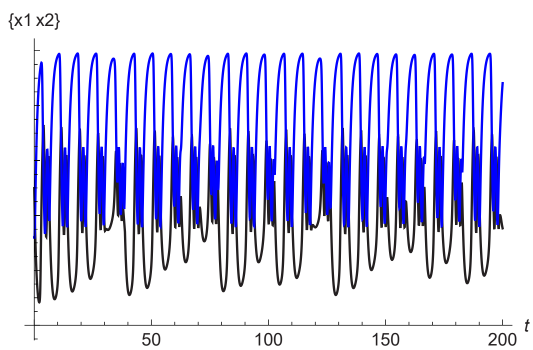

Figure 63: The graphs of solutions $(x_1(t), x_2(t))$ of the system (41) with the regulatory matrix (55), $k_1 = 0.5$ and $k_2 = 1.815$, $w_{14} = -1$.

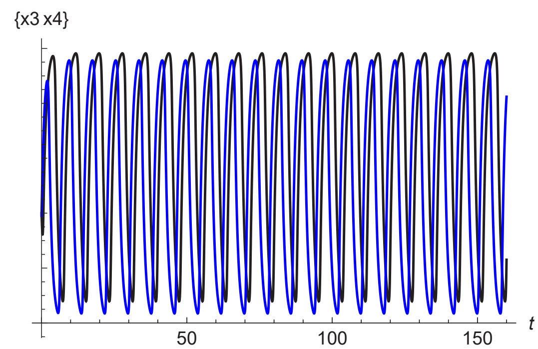

Figure 64: The graphs of solutions $(x_3(t), x_4(t))$ of the system (41) with the regulatory matrix (55), $k_1 = 0.5$ and $k_2 = 1.815$, $w_{14} = -1$.

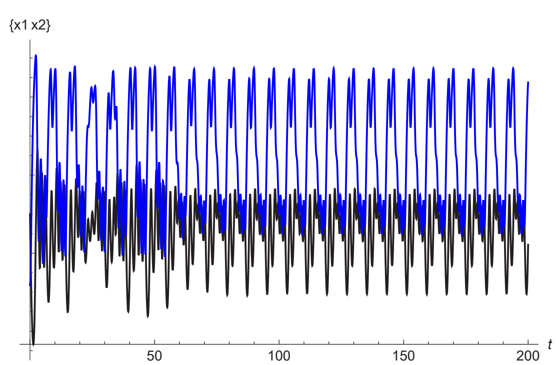

Figure 65: The graphs of solutions $(x_1(t), x_2(t))$ of the system (41) with the regulatory matrix (55), $k_1 = 0.5$ and $k_2 = 1.815$, $w_{14} = -0.7$.

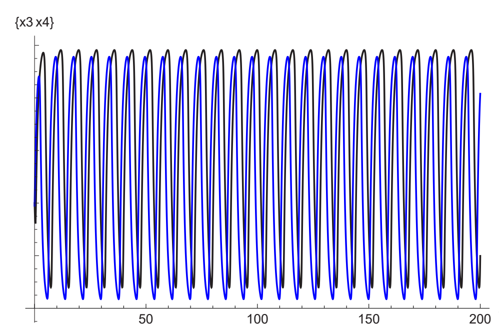

Figure 66: The graphs of solutions $(x_3(t), x_4(t))$ of the system (41) with the regulatory matrix (55), $k_1 = 0.5$ and $k_2 = 1.815$, $w_{14} = -0.7$.

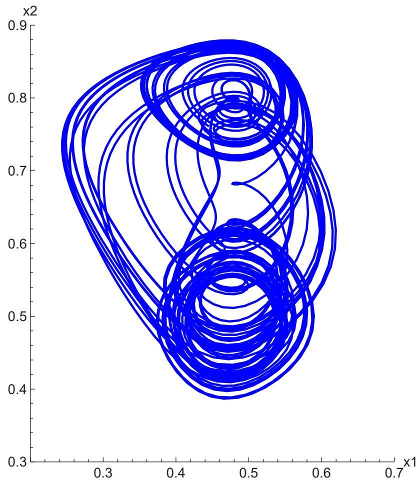

Figure 67: The projection of 4D trajectories to 2D subspace $(x_1(t), x_2(t)), w_{14} = -0.7$.

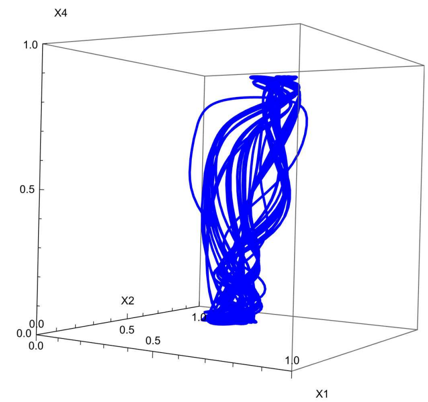

Figure 68: The projection of 4D trajectories to 3D subspace $(x_1(t), x_2(t), x_4(t)), w_{14} = -0.7$.

The dynamics of Lyapunov exponents are shown in Figure 69 and Figure 70.

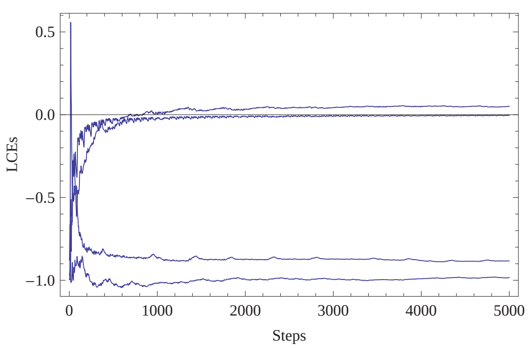

Figure 69: The dynamics of Lyapunov exponents of the system (41) with the regulatory matrix (55), $k_1 = 0.5$ and $k_2 = 1.815$, $w_{14} = -1$.

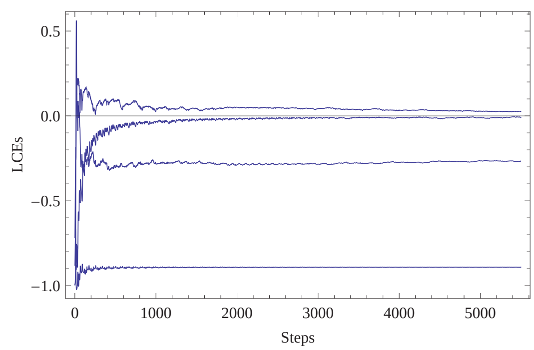

Figure 70: The dynamics of Lyapunov exponents of the system (41) with the regulatory matrix (55), $k_1 = 0.5$ and $k_2 = 1.815$, $w_{14} = -0.7$.

6.7 4D system from 3D and 1D systems

Let us return to the GRN system (41).

Consider

$$W = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & k \end{pmatrix} \tag{56}$$

and $\mu_1 = \mu_3 = \mu_4 = 5$, $\mu_2 = 15$, $v_1 = v_2 = v_3 = v_4 = 1$, $\theta_1 = 1.2$, $\theta_2 = 0.5$, $\theta_3 = -0.6$, $\theta_4 = -0.2$. The system is uncoupled, consisting of one independent 3D system and one

1D system. The 3D system has one critical point. The periodic solution is depicted in Figure 37.

Table 6. Results of calculations for the system (41) with regulatory matrix (56), changing the parameter k.

k	x^*	y^*	z^*	m^*	λ_1	λ_2	$\lambda_{3,4}$
-4	0.537	0.999	0.346	0.141	-3.42	-0.99	0.188 ± 2.372
-3	0.537	0.999	0.346	0.172	-3.13	-0.99	0.188 ± 2.372
-2	0.537	0.999	0.346	0.224	-2.74	-0.99	0.188 ± 2.372
-1	0.537	0.999	0.346	0.336	-2.12	-0.99	0.188 ± 2.372
-0.1	0.537	0.999	0.346	0.661	-1.11	-0.99	0.188 ± 2.372
0.1	0.537	0.999	0.346	0.802	-0.99	-0.92	0.188 ± 2.372
1	0.537	0.999	0.346	0.998	-0.99	-0.988	0.188 ± 2.372
2	0.537	0.999	0.346	0.999	-0.99	-0.99	0.188 ± 2.372
4	0.537	0.999	0.346	0.999	-0.999	-0.999	0.188 ± 2.372

There is one periodic solution.

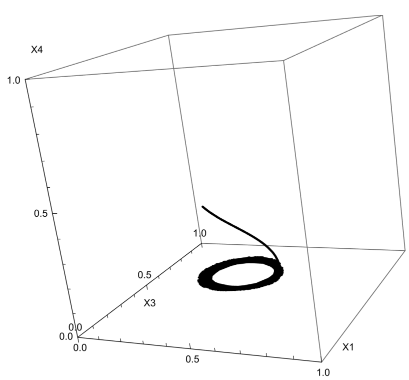

Figure 71: The projection of 4D trajectories to 3D subspace (x_1, x_3, x_4), $k = -4$

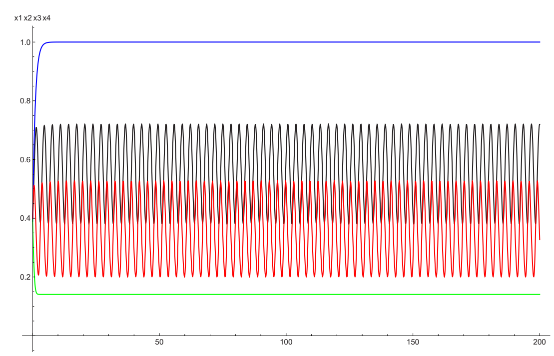

Figure 72: The graphs of periodic solutions $(x_1(t), x_2(t), x_3(t), x_4(t))$ of the system (41) with the regulatory matrix (56), $k = -4$.

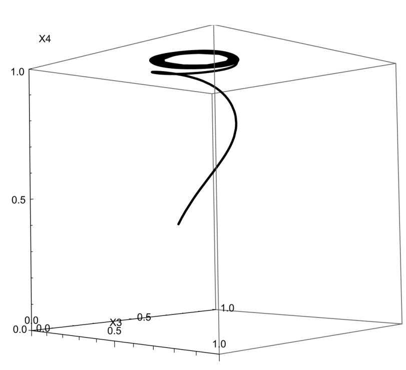

Figure 73: The projection of 4D trajectories to 3D subspace (x_1, x_3, x_4), $k = 4$.

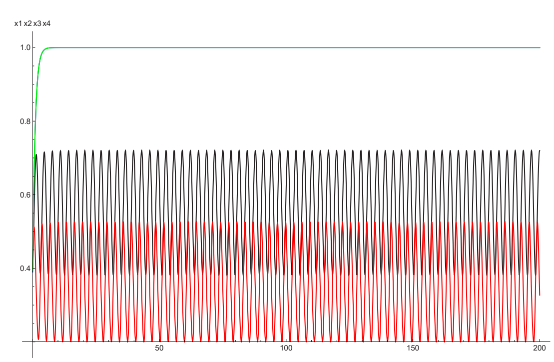

Figure 74: The graphs of periodic solutions $(x_1(t), x_2(t), x_3(t), x_4(t))$ of the system (41) with the regulatory matrix (56), $k = 4$.

6.8 Examples

In below examples the 4D GRN system (41) is considered.

Example 1. Now fill in all zero elements of the regulatory matrix (55) with values 0.1, so the regulatory matrix is

$$W = \begin{pmatrix} 0.5 & 2 & 0.1 & 0.1 \\ -2 & 0.5 & 0.1 & 0.1 \\ 0.1 & 0.1 & 1.815 & 2 \\ 0.1 & 0.1 & -2 & 1.815 \end{pmatrix}. \tag{57}$$

The parameters and the initial conditions are the same. There is exactly one critical point. The standard linearization analysis provides the characteristic numbers $\lambda_{1,2} = 0.2159 \pm 4.939i$ and $\lambda_{3,4} = 3.375 \pm 4.789i$.

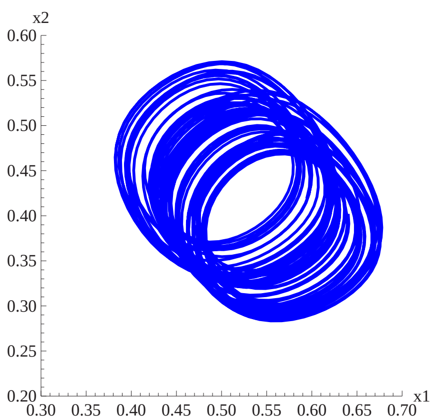

Figure 75: The projection of 4D trajectories to 2D subspace (x_1, x_2).

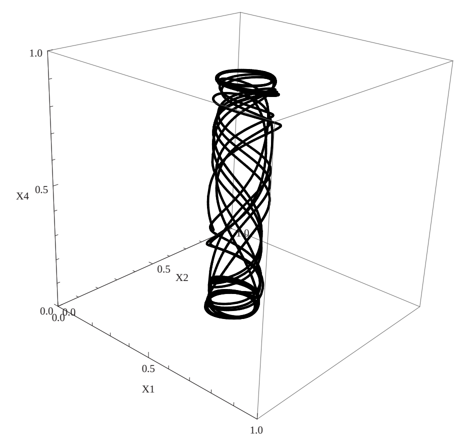

Figure 76: The projection of 4D trajectories to 3D subspace (x_1, x_2, x_4).

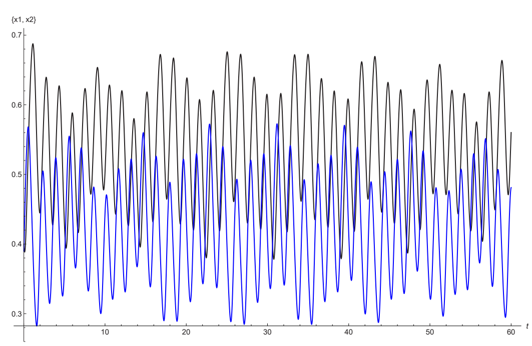

Figure 77: The graphs of solutions $(x_1(t), x_2(t))$ of the system (41) with the regulatory matrix (57).

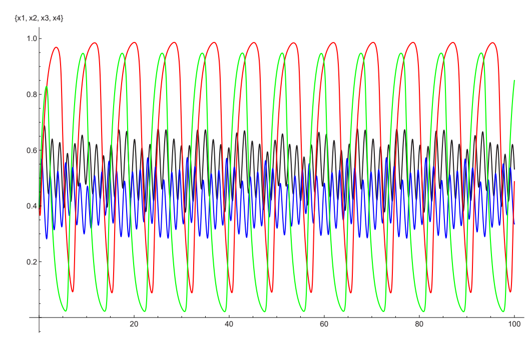

Figure 78: The graphs of solutions $(x_1(t), x_2(t), x_3(t), x_4(t))$ of the system (41) with the regulatory matrix (57).

Solutions are visually chaotic, in fact, they are not chaotic.

The dynamics of Lyapunov exponents are shown in Figure 79.

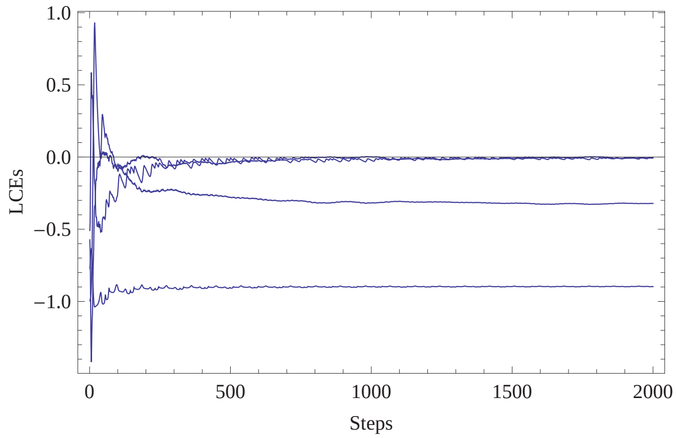

Figure 79: $LE_1 = 0, LE_2 = -0.007, LE_3 = -0.32, LE_4 = -0.90$

Example 2. The regulatory matrix is

$$W = \begin{pmatrix} 0.9 & 1.9 & 0.3 & 1 \\ -2 & 1 & -0.1 & 0.1 \\ 0.1 & 0.1 & 1 & 2 \\ 0.3 & 0.3 & -1.8 & 0.8 \end{pmatrix} \tag{58}$$

and the parameters $v_1 = v_2 = v_3 = v_4 = 1$, $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 9$, $\theta_1 = 1.2$, $\theta_2 = -0.6$, $\theta_3 = 1.2$, $\theta_4 = -0.6$.

The initial conditions are

$$x_1(0) = 0.46;\ x_2(0) = 0.16;\ x_3(0) = 0.62;\ x_4(0) = 0.29.$$

The critical point is $(0.8863; 0.0771; 0.6778; 0.2421)$. The standard linearization analysis provides the characteristic numbers $\lambda_1 = -0.998266$, $\lambda_2 = 0.0007695$ and $\lambda_{3,4} = 0.704708 \pm 3.33305i$. The type of the critical point is an unstable saddle-focus.

The critical point is $(0.9088; 0.6949; 0.6785; 0.2362)$. The standard linearization analysis provides the characteristic numbers $\lambda_1 = -0.998927$, $\lambda_2 = -0.000699007$ and $\lambda_{3,4} = 0.721755 \pm 3.35263i$. The type of the critical point is an unstable node-focus.

The critical point is $(0.4552; 0.1567; 0.6190; 0.2869)$. The standard linearization analysis provides the characteristic numbers $\lambda_{1,2} = 0.103594 \pm 3.13907i$ and $\lambda_{3,4} = 1.2932 \pm 3.64712i$ The type of the critical point is an unstable focus-focus. The projection of this attractor on two-dimensional subspace (x_1, x_2) is in the figure below.

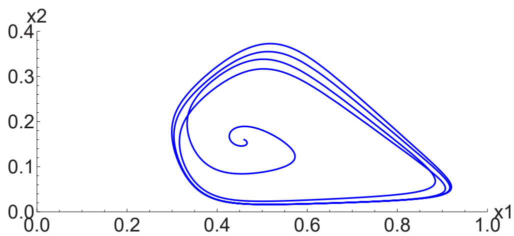

Figure 80: The projection of 4D trajectories to 2D subspace (x_1, x_2).

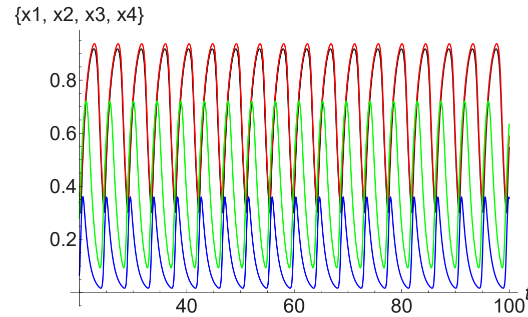

Figure 81: The graphs of periodic solutions $(x_1(t), x_2(t), x_3(t), x_4(t))$ of the system (41) with the regulatory matrix (58).

Example 3. The regulatory matrix is

$$W = \begin{pmatrix} 0.8 & 2 & -0.8 & 0.5 \\ -2 & 0.3 & 0.4 & -0.7 \\ -0.5 & 0.2 & 1.8 & 2 \\ 0.8 & -0.7 & -2 & 1.8 \end{pmatrix} \tag{59}$$

and the parameters $v_1 = v_2 = v_3 = v_4 = 1$, $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 10$ and θ_i, where $i = 1, 2, 3, 4$ are calculated as

$$\begin{cases} \theta_1 = \dfrac{w_{11} + w_{12} + w_{13} + w_{14}}{2}, \\ \theta_2 = \dfrac{w_{21} + w_{22} + w_{23} + w_{24}}{2}, \\ \theta_3 = \dfrac{w_{31} + w_{32} + w_{33} + w_{34}}{2}, \\ \theta_4 = \dfrac{w_{41} + w_{42} + w_{43} + w_{44}}{2}. \end{cases}$$

$\theta_1 = 1.25$, $\theta_2 = -1$, $\theta_3 = 1.75$, $\theta_4 = -0.05$.

The initial conditions are

$$x_1(0) = 0.4;\ x_2(0) = 0.6;\ x_3(0) = 0.39;\ x_4(0) = 0.38. \tag{60}$$

The critical point is $(0.5; 0.5; 0.5; 0.5)$. The standard linearization analysis provides the characteristic numbers $\lambda_{1,2} = -0.44 \pm 4.603i$ and $\lambda_{3,4} = 4.33 \pm 5.135i$. The type of the critical point is an unstable focus-focus.

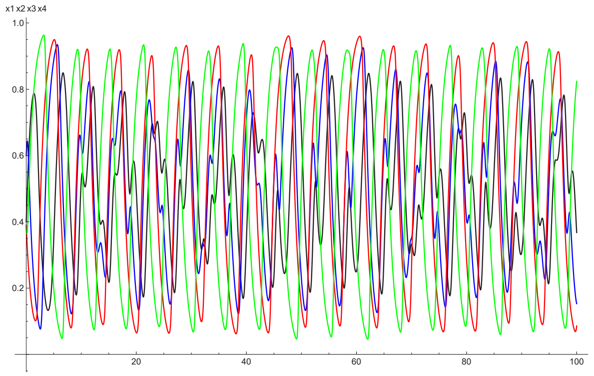

Figure 82: The graphs of solutions $(x_1(t), x_2(t), x_3(t), x_4(t))$ of the system (41) with the regulatory matrix (59).

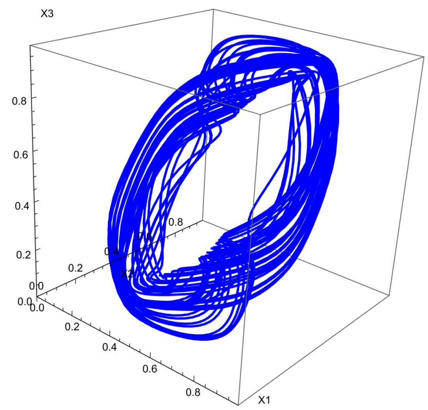

Figure 83: The projection of 4D trajectories to 3D subspace (x_1, x_2, x_3).

The dynamics of Lyapunov exponents are shown in Figure 84.

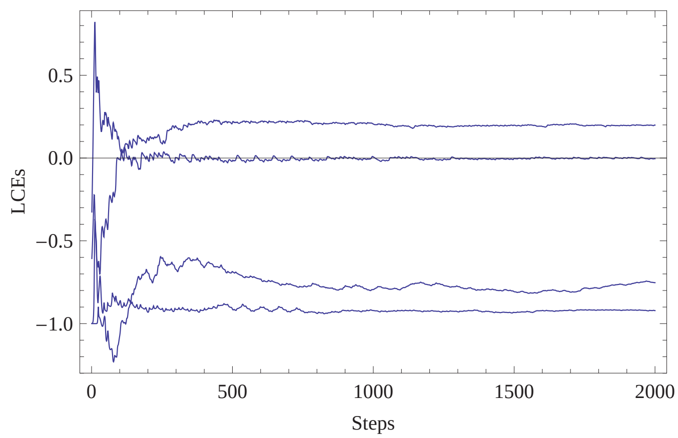

Figure 84: $LE_1 = 0.20, LE_2 = 0, LE_3 = -0.75, LE_4 = -0.92$

$LE_1, LE_2, LE_3, LE_4 = (+, 0, -, -)$ is a self-excited chaotic attractor. The behavior of the system (41) with regulatory matrix (59) and initial conditions (60) is chaotic.

6.9 Conclusions for four-dimensional systems

The following are true for the system (41):

- the four-dimensional system (41) can have attractors of various kinds;
- the four-dimensional system (41) can have several stable periodic solutions, which serve as attractors;
- the irregular behavior of solutions near the chaotic attractor is possible. It can appear in a very small range of parameters;
- the self-excited chaotic attractor is possible.

7 Five-dimensional (5D) systems

The system of ODE consisting of five equations is

$$\begin{cases} \dfrac{dx_1}{dt} = f_1(w_{11}x_1 + \ldots + w_{15}x_5) - v_1x_1, \\ \\ \dfrac{dx_2}{dt} = f_2(w_{21}x_1 + \ldots + w_{25}x_5) - v_2x_2, \\ \\ \qquad\qquad \cdots\cdots\cdots \\ \\ \dfrac{dx_5}{dt} = f_5(w_{51}x_1 + \ldots w_{55}x_5) - v_5x_5. \end{cases} \tag{61}$$

All processes occurring in the body (biochemical and physiological) are carried out due to the coordinated expression of various groups of genes. Each such group forms the basis of a specific gene network responsible for the performance of a specific function of a cell, organ, or organism. Beneath the gene network, this refers to the set of coordinately expressed genes, their protein products, and the relationships between them. The higher the dimension of the system, the more realistic the mathematical model is.

7.1 Artificial Neural Networks

Consider the system

$$\begin{cases} x_1' = \tanh(x_4 - x_2) - bx_1, \\ x_2' = \tanh(x_1 + x_4) - bx_2, \\ x_3' = \tanh(x_1 + x_2 - x_4) - bx_3, \\ x_4' = \tanh(x_3 - x_2) - bx_4, \\ x_5' = \tanh(x_1 + x_4 - x_5) - bx_5 \end{cases} \tag{62}$$

and $b = 0.043$.

The initial conditions are

$$x_1(0) = 1.2;\ x_2(0) = 0.4;\ x_3(0) = 1.2;\ x_4(0) = -1;\ x_5(0) = -1.$$

The graph of the system (62) is depicted in Figure 85.

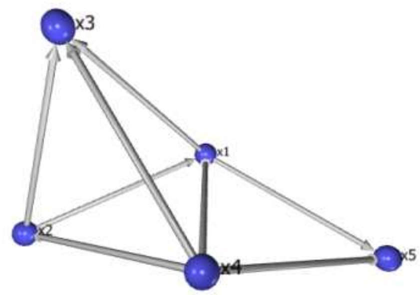

Figure 85: The graph, corresponding to the system (62).

This system has an attractor as shown in Figure 86 and Figure 87. The irregular behavior of three solutions can be seen in Figure 88 and Figure 89.

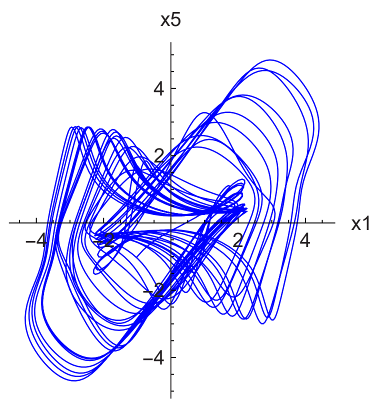

Figure 86: The projection of the attractor on 2D subspace $(x_4(t), x_5(t))$

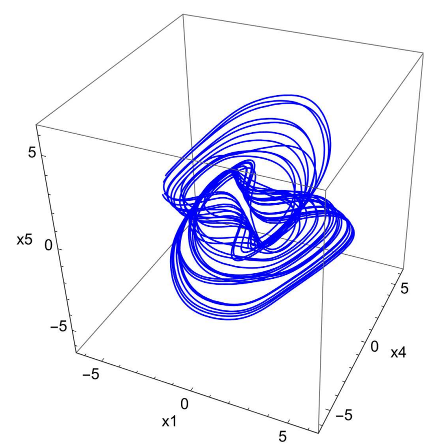

Figure 87: The projection of the attractor on 3D subspace $(x_1(t), x_4(t), x_5(t))$

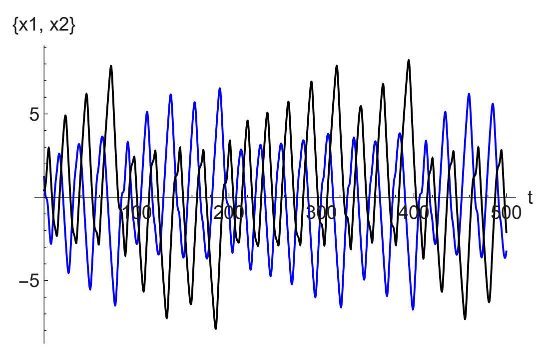

Figure 88: The graphs of solutions $(x_1(t), x_2(t))$ of the system (62).

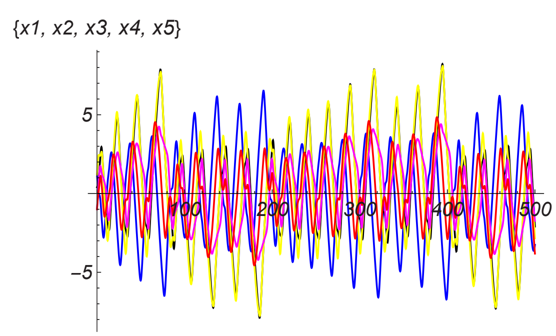

Figure 89: The graphs of solutions $(x_1(t), x_2(t), x_3(t), x_4(t), x_5(t))$ of the system (62).

7.2 5D system from 2D and 3D systems

Example 1. Consider the five-dimensional system (61). Let the regulatory matrix be

$$W = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{63}$$

and $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = 5$, $v_1 = v_2 = v_3 = v_4 = v_5 = 1$, $\theta_1 = 1.5$, $\theta_2 = \theta_3 = 1$ and $\theta_4 = \theta_5 = 0.5$.

The initial conditions are

$$x_1(0) = 0.4;\ x_2(0) = 0.395;\ x_3(0) = 0.4;\ x_4(0) = 0.395;\ x_5(0) = 0.1.$$

This system consists of one three-dimensional system and one two-dimensional system.

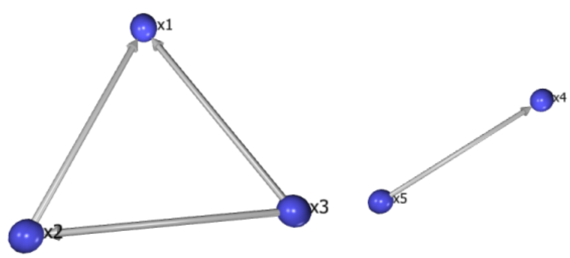

Figure 90: The graph, corresponding to the case of the regulatory matrix (63).

This system is uncoupled and has one critical point $(0.5, 0.5, 0.5, 0.5, 0.5)$. The solution of the system (61) with the regulatory matrix (63) is stable.

Example 2. Consider the five-dimensional system (61). Let the regulatory matrix be

$$W = \begin{pmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 2 \\ 0 & 0 & 0 & -2 & 0.5 \end{pmatrix} \tag{64}$$

and $\mu_1 = \mu_3 = 5$, $\mu_2 = 15$, $\mu_4 = \mu_5 = 10$, $v_1 = v_2 = v_3 = v_4 = v_5 = 1$, $\theta_1 = 1.2$, $\theta_2 = 0.5$, $\theta_3 = -0.6$, $\theta_4 = 1.2$, $\theta_5 = -0.7$.

The initial conditions are

$$x_1(0) = 0.5;\ x_2(0) = 0;\ x_3(0) = 0.3;\ x_4(0) = 0.5;\ x_5(0) = 0.3.$$

This system consists of one three-dimensional system, which has a periodic solution depicted in Figure 34, and one two-dimensional system, which has a periodic solution depicted in Figure 7. This system is uncoupled and has three critical points. The solution of the system (61) with regulatory matrix (64) is periodic.

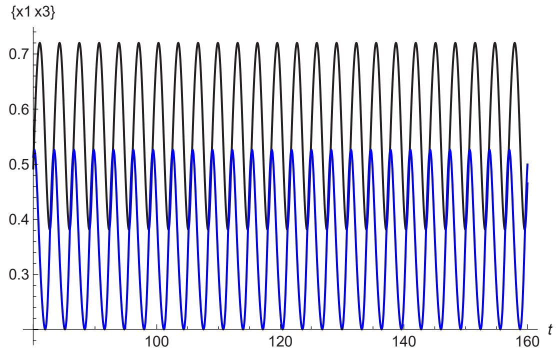

Figure 91: The graphs of solutions $x_i(t)$, $i = 1, 3$ of the system (61) with the regulatory matrix (64).

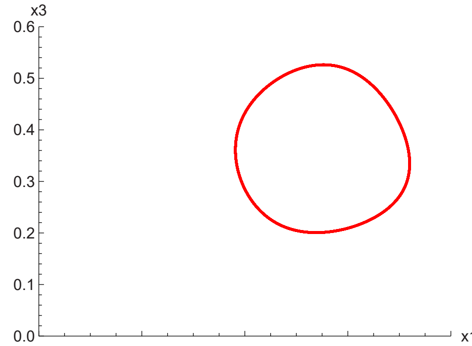

Figure 92: The projection of 5D trajectories to 2D subspace (x_1, x_3).

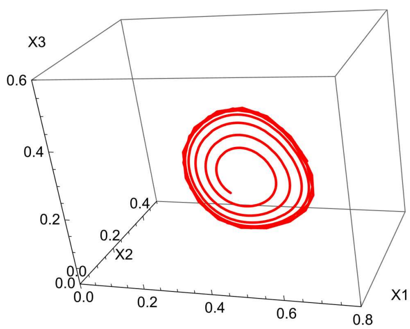

Figure 93: The projection of 5D trajectories to 3D subspace (x_1, x_2, x_3).

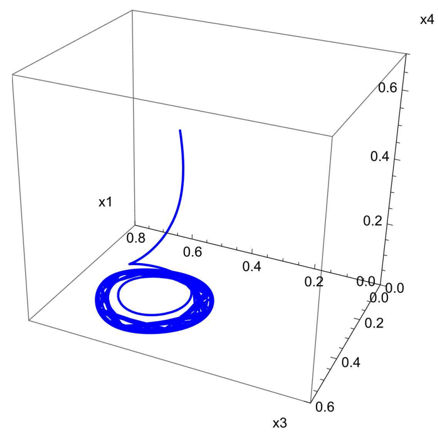

Figure 94: The projection of 5D trajectories to 3D subspace (x_1, x_3, x_4).

Example 3. Consider the five-dimensional system (61). Let the regulatory matrix be (64). All zero elements we change to 0.1. Other parameters remain the same.

The initial conditions are

$$x_1(0) = 0.392;\ x_2(0) = 0;\ x_3(0) = 0.397;\ x_4(0) = 0.35;\ x_5(0) = 0.3.$$

This system is coupled. The solution of this system is still periodic.

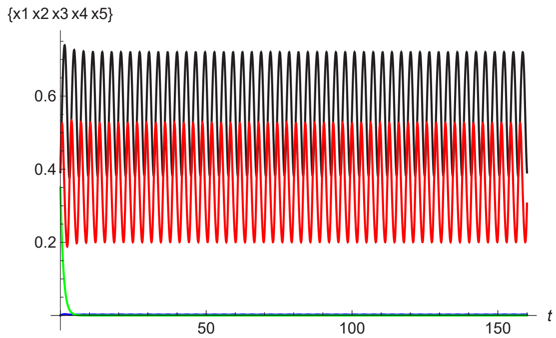

Figure 95: The graphs of solutions $x_i(t)$, $i = 1, 2, 3, 4, 5$ of the system (61) with the regulatory matrix (64), all zero elements equal 0.1.

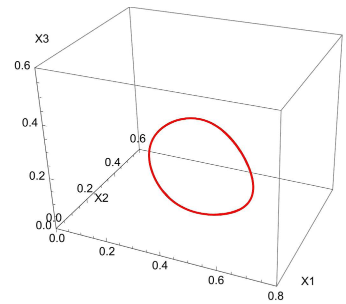

Figure 96: The projection of 5D trajectories to 3D subspace (x_1, x_2, x_3)

7.3 Conclusions for five-dimensional systems

The following are true for the system (61):

- the five-dimensional system (61) can have attractors of various kinds;
- the five-dimensional system (61) can have several stable periodic solutions, which serve as attractors;
- the irregular behavior of solutions near the attractor in Artificial Neural Networks is possible.

8 Six-dimensional (6D) systems

The system of ODE consisting of six equations is

$$\begin{cases} \dfrac{dx_1}{dt} = f_1(w_{11}x_1 + \ldots + w_{16}x_6) - v_1 x_1, \\ \dfrac{dx_2}{dt} = f_2(w_{21}x_1 + \ldots + w_{26}x_6) - v_2 x_2, \\ \qquad \ldots\ldots\ldots \\ \dfrac{dx_6}{dt} = f_6(w_{61}x_1 + \ldots w_{66}x_6) - v_6 x_6. \end{cases} \tag{65}$$

Similar systems of dimensionality two, three, four and of arbitrary dimensionality [58],[63] appear in various contexts describing neuronal networks [14],[13], genetic networks [89], telecommunications networks [29] and more. This type models can reflect an evolution in time t of a network. Networks management and control are possible by changing system parameters [70], [4].

8.1 Artificial Neural Networks

Consider the system

$$\begin{cases} x_1' = \tanh(x_4 - x_2) - bx_1, \\ x_2' = \tanh(x_1 + x_4) - bx_2, \\ x_3' = \tanh(x_1 + x_2 - x_4) - bx_3, \\ x_4' = \tanh(x_3 - x_2) - bx_4, \\ x_5' = \tanh(x_1 + x_4 - x_5 + x_6) - bx_5, \\ x_6' = \tanh(x_1 + x_4) - bx_6 \end{cases} \tag{66}$$

and $b = 0.043$.

The initial conditions are

$$x_1(0) = 1.2;\ x_2(0) = 0.4;\ x_3(0) = 1.2;\ x_4(0) = -1;\ x_5(0) = -1;\ x_6(0) = -1.$$

The graph of the system (66) is depicted in Figure 97.

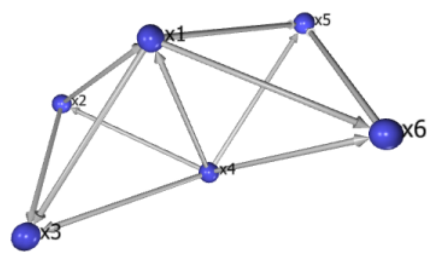

Figure 97: The graph, corresponding to the case of the regulatory matrix (66).

This system has an attractor as shown in Figure 98 and Figure 99. The irregular behavior of solutions can be seen in Figure 100 and Figure 101.

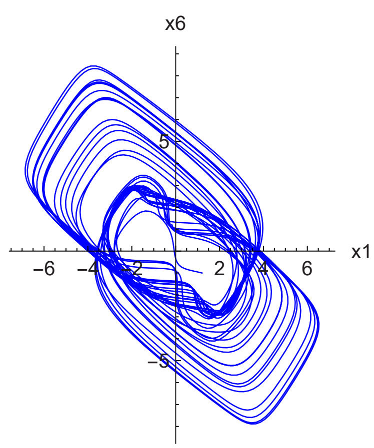

Figure 98: The projection of the attractor on 2D subspace $(x_1(t), x_6(t))$

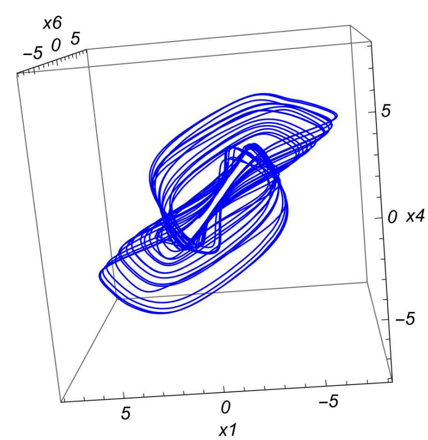

Figure 99: The projection of the attractor on 3D subspace $(x_1(t), x_4(t), x_6(t))$

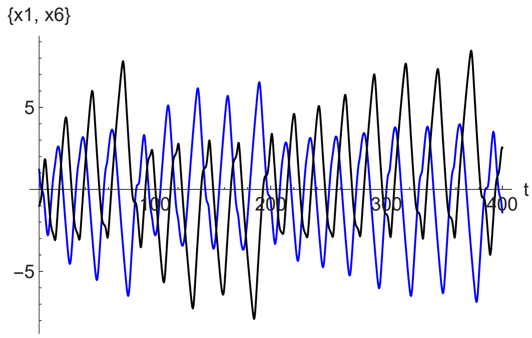

Figure 100: The graphs of solutions $(x_1(t), x_6(t))$ of the system (66).

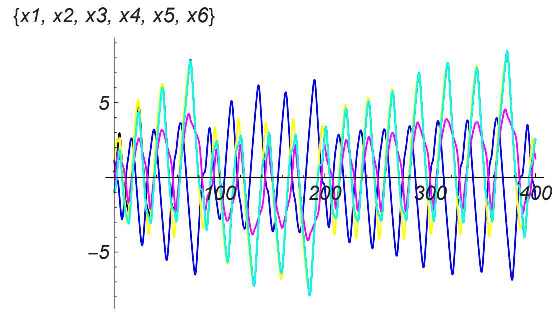

Figure 101: The graphs of solutions $(x_1(t), x_2(t), x_3(t), x_4(t), x_5(t), x_6(t))$ of the system (66).

This system can exhibit three types of dynamics. Solutions can approach a static equilibrium and thereafter remain forever. It means that the human brain is dead. Solutions can be periodic or quasi-periodic. In this case, the human brain does not develop, which means it does not have the ability to think creatively. Solutions can be chaotic, which is arguably the most healthy state for a natural network, especially if it is only weakly chaotic so that it retains some memory but can explore a vastly greater state space. Weakly chaotic networks exhibit the complex behaviour that we normally associate with intelligent living systems [82].

8.2 6D system from 2D systems

Example 1. Consider the six-dimensional system (65). Let the regulatory matrix be

$$W = \begin{pmatrix} 0.5 & 2 & 0 & 0 & 0 & 0 \\ -2 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 2 \\ 0 & 0 & 0 & 0 & -2 & 0.5 \end{pmatrix} \tag{67}$$

and $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu_6 = 10$, $v_1 = v_2 = v_3 = v_4 = v_5 = v_6 = 1$, $\theta_1 = \theta_3 = \theta_5 = 1.2$ and $\theta_2 = \theta_4 = \theta_6 = -0.7$.
This system consists of three independent two-dimensional systems, which have an attractor depicted in Figure 7. The resulting attractor is a product of three two-dimensional ones and is, therefore, periodic. A trial solution with the initial values

$$x_1(0) = 0.68,\ x_2(0) = 0.3,\ x_3(0) = 0.1,\ x_4(0) = 0.6,\ x_5(0) = 0.2,\ x_6(0) = 0.1 \tag{68}$$

was used to reveal the six-dimensional attractor.

Change now two elements at the right upper (w_{16}) and left lower (w_{61}) corners. Let $w_{16} = w_{61} = 0.5$. The six-dimensional system (65) is not yet uncoupled. The trial solution still tends to periodic attractor (a different one), however. The graphs of all six solutions $x_i(t)$ are depicted in Figure 102 and Figure 103.

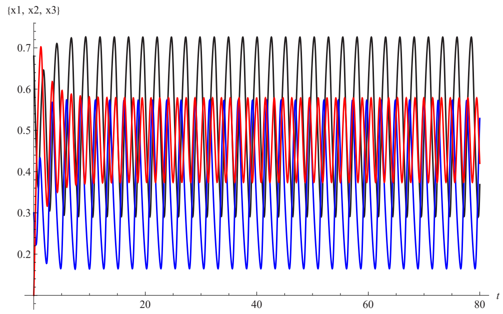

Figure 102: The graphs of $x_i(t)$, $i = 1, 2, 3$, of the system (65) with the regulatory matrix (67), $w_{16} = w_{61} = 0.5$.

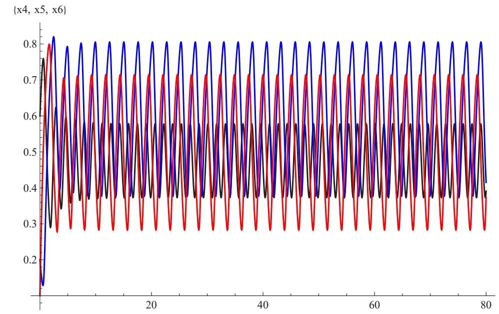

Figure 103: The graphs of $x_i(t)$, $i = 4, 5, 6$, of the system (65) with the regulatory matrix (67), $w_{16} = w_{61} = 0.5$.

8.3 6D system from 3D systems

Consider the six-dimensional system (65). Our intent now is to create a six-dimensional attractor from three-dimensional ones.

Example 1. Consider the six-dimensional system (65) with the regulatory matrix

$$W = \begin{pmatrix} k_1 & 0 & -1 & 0 & 0 & 0 \\ -1 & k_1 & 0 & 0 & 0 & 0 \\ 0 & -1 & k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_2 & 0 & -1 \\ 0 & 0 & 0 & -1 & k_2 & 0 \\ 0 & 0 & 0 & 0 & -1 & k_2 \end{pmatrix}, \tag{69}$$

where $k_1 = k_2 = 1$, $\mu_i = 5$, $\theta_i = \frac{k-1}{2}$.

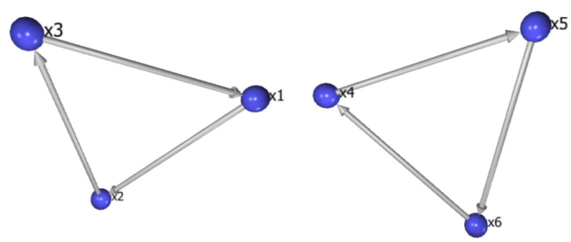

Figure 104: The graph, corresponding to the case of the regulatory matrix (69).

The initial conditions are

$$x_1(0) = 0.046;\ x_2(0) = 0.8;\ x_3(0) = 0.3;\ x_4(0) = 0.7;\ x_5(0) = 0.8;\ x_6(0) = 0.2.$$

The 6D system has an attractor in the form of a periodic solution generated by a three-dimensional periodic solution shown in Figure 30 and Figure 32. The projections of this periodic attractor onto three-dimensional subspaces are shown in Figure 105 and Figure 106.

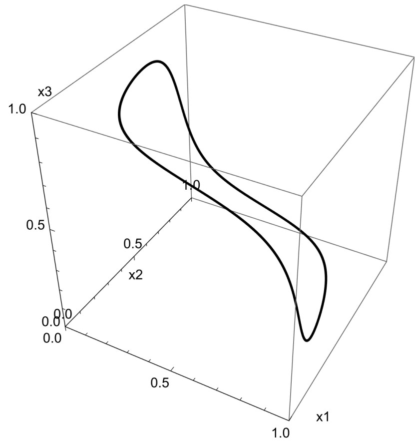

Figure 105: The projections of 6D trajectories to 3D subspace (x_1, x_2, x_3).

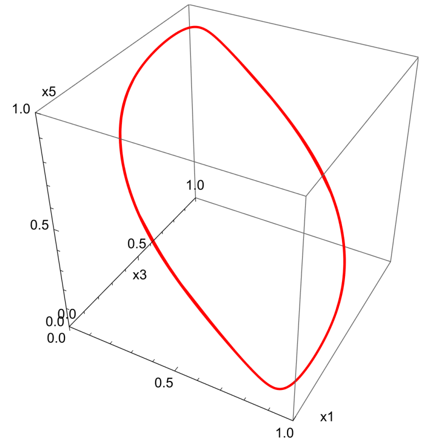

Figure 106: The projections of 6D trajectories to 3D subspace (x_1, x_3, x_5).

Consider the six-dimensional system (65) with the regulatory matrix (69), where $k_1 = 1,\ k_2 = 0.5,\ \mu_i = 5,\ \theta_i = \frac{k-1}{2}$.

The initial conditions are

$$x_1(0) = 0;\ x_2(0) = 0.4;\ x_3(0) = 0.1;\ x_4(0) = 0.2;\ x_5(0) = 0.1;\ x_6(0) = 0.1.$$

The projections of this periodic attractor onto three-dimensional subspaces are shown in Figure 107 and Figure 108.

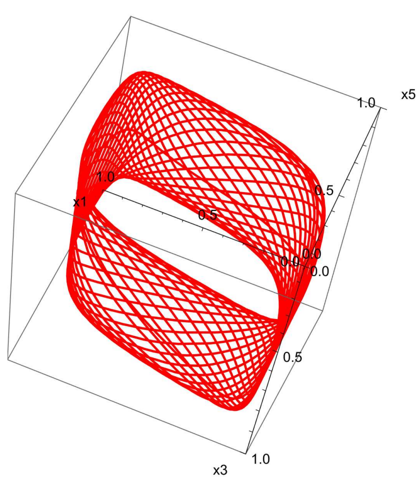

Figure 107: The projections of 6D trajectories to 3D subspace (x_1, x_3, x_5).

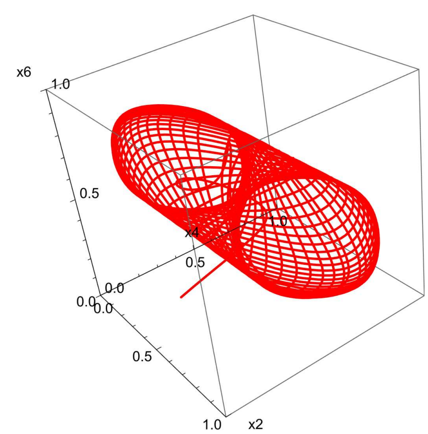

Figure 108: The projections of 6D trajectories to 3D subspace (x_2, x_4, x_6).

The respective six-dimensional system was studied in [68].

Example 2. We take the three-dimensional system (19) with the regulatory matrix (39), set of parameters (38) and initial conditions (40). The irregular behavior of three solutions can be seen in Figure 45.

Consider the six-dimensional system (65) with the regulatory matrix

$$W = \begin{pmatrix} 0 & 1 & -5.64 & 0 & 0 & 0 \\ 1 & 0 & 0.1 & 0 & 0 & 0 \\ 1 & 0.02 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -5.64 \\ 0 & 0 & 0 & 1 & 0 & 0.1 \\ 0.5 & 0 & 0 & 1 & 0.02 & 0 \end{pmatrix} \tag{70}$$

and

$$\mu_1 = \mu_2 = \mu_4 = \mu_5 = 7, \mu_3 = \mu_6 = 13, v_1 = v_4 = 0.65, v_2 = v_5 = 0.42, v_3 = v_6 = 0.1,$$

$$\theta_1 = \theta_4 = 0.5, \theta_3 = \theta_5 = 0.3, \theta_3 = \theta_6 = 0.7.$$

The initial conditions are

$$x_1(1) = 0.68;\ x_2(1) = 0.45;\ x_3(1) = 0.15;\ x_4(1) = 0.68;\ x_5(1) = 0.45;\ x_6(1) = 0.15.$$

It would be uncoupled if the element w_{61} be zero. Then we would have a six-dimensional attractor which is the product of two identical three-dimensional attractors as in Figure 44. But w_{61} is set to 0.5. The six-dimensional system is coupled now. The new attractor exists and some of the three-dimensional projections are depicted in Figure 109 and Figure 110.

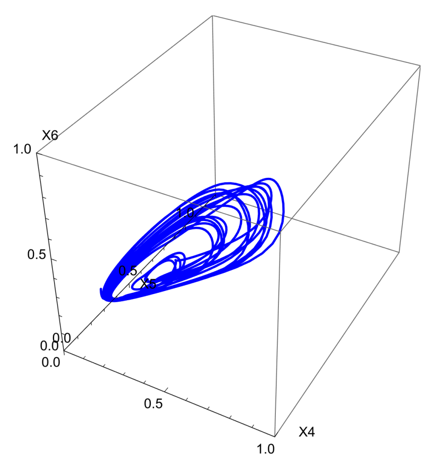

Figure 109: The projections of 6D trajectories to 3D subspace (x_4, x_5, x_6).

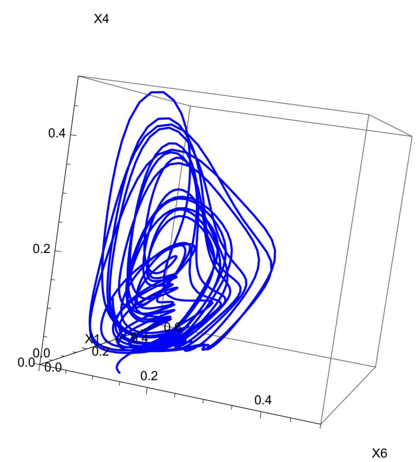

Figure 110: The projections of 6D trajectories to 3D subspace (x_1, x_4, x_6).

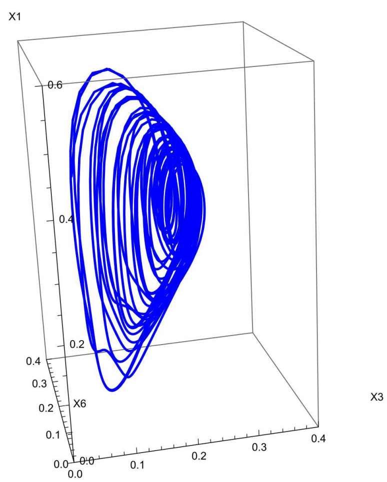

Figure 111: The projections of 6D trajectories to 3D subspace (x_1, x_3, x_6).

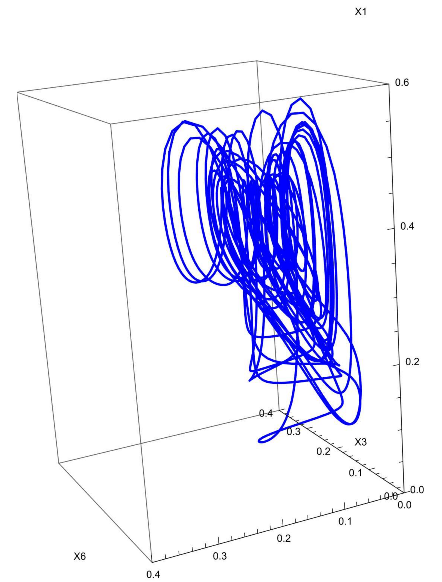

Figure 112: The projections of 6D trajectories to 3D subspace (x_1, x_3, x_6).

The solutions for system (65) with the matrix (70) are depicted in Figure 113 and Figure 114.

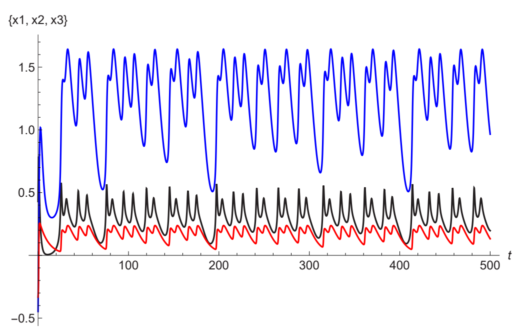

Figure 113: The graphs of solutions $x_i(t)$, $i = 1, 2, 3$, of the system (65) with the regulatory matrix (70), $w_{61} = 0.5$.

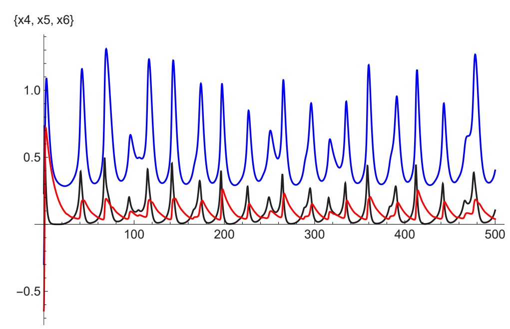

Figure 114: The graphs of solutions $x_i(t)$, $i = 4, 5, 6$, of the system (65) with the regulatory matrix (70), $w_{61} = 0.5$.

The graphs of solutions have irregular forms. They are different in Figure 113 and Figure 114 because of the non-zero element w_{61}.

8.4 Conclusions for six-dimensional systems

The following are true for the system (65):

- the six-dimensional system (41) can have attractors of various kinds;
- the six-dimensional system (41) can have several stable periodic solutions, which serve as attractors;
- the irregular behavior of solutions near the attractor is possible.

9 Sixty-dimensional (60D) systems

The network taken for the study is a realistic biological network, “T cells in large granular lymphocyte leukemia associated with blood cancer”. A network model considered in [10],[89], contains 60 nodes and 195 regulatory edges. It was found in [89] that this network has three attractors, of which two correspond to two distinct cancerous states (denoted as C_1 and C_1) and one is associated with the normal state (denoted as N). The proper selection of the respective forty-eight parameters can drive the system to the normal state. The existence of needed parameter perturbation was acknowledged. The attractor network was considered and the main proposition was to arrange experimental adjustment of parameters in order to achieve the required goal.

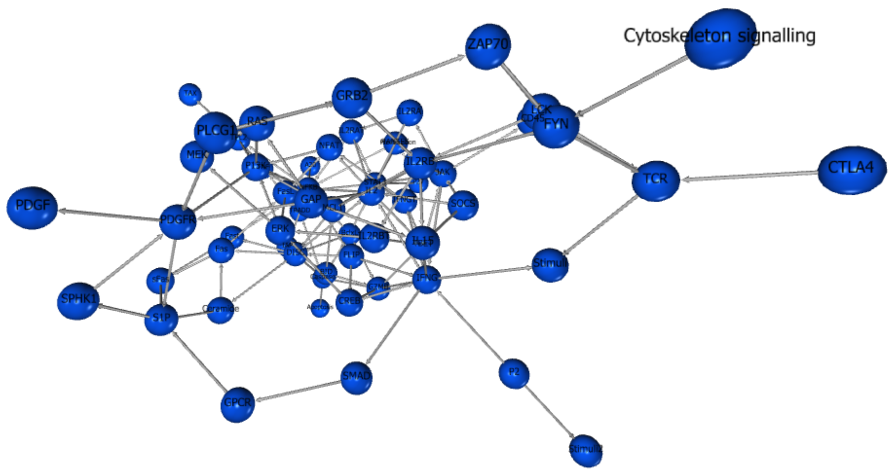

Figure 115: The graph of matrix 60×60.

To obtain this graph, the “Graphia” program was used. The matrix (116) was written in the program “Microsoft Excel”.

	a1 Cytoskeleton signalling	a2 FYN	a3 ZAP70	a4 GRB2	a5 PLCG1	a6 PDGF	a7 PDGFR	a8 SPHK1	a9 S1P	a10 sFas	a11 Ceramide	a12 GPCR	a13 SMAD	a14 RAS	a15 GAP	a16 MEK	a17 ERK	a18 P13K	a19 Fas	a20 Fast	a21 TAX	a22 TPL2	a23 DISC	a24 FasL	a25 NFAT	a26 CREB	a27 IL2RBT	a28 TBET	a29 IL2RB	a30 IL15
w1 Cytoskeleton signalling																														
w2 FYN	1		-1																											
w3 ZAP70				1																										
w4 GRB2					1									1																
w5 PLCG1														1																
w6 PDGF							1																							
w7 PDGFR					1			1							1			1												
w8 SPHK1									1																					
w9 S1P							1			1	-1	1																		
w10 sFas																			-1											
w11 Ceramide									-1														1							
w12 GPCR													1																	
w13 SMAD																														
w14 RAS															1	1		1												
w15 GAP														-1	1															
w16 MEK																	1													
w17 ERK																								1		1	1			
w18 P13K																	1					1			1					
w19 Fas											1												1							
w20 Fast										1									1				1							
w21 TAX																						1								
w22 TPL2																														
w23 DISC																														
w24 FasL																			1											
w25 NFAT																								1						
w26 CREB																														
w27 IL2RBT																													1	
w28 TBET																											1	1		
w29 IL2RB		1		1																										
w30 IL15															-1														1	
w31 IFNG																										1				
w32 FLIP																							-1							
w33 TNF																						1								
w34 MCL1																														
w35 NFKB																				1				1						
w36 A20																														
w37 TRADD																														
w38 IAP																														
w39 Caspase																														
w40 BclxL																														
w41 GZMB																														
w42 IL2															-1								1						1	
w43 LCK			1																											
w44 Stimuli																														
w45 TCR		1																												
w46 CTLA4																														
w47 JAK																												1		
w48 SOCS																														
w49 IFNGT																														
w50 STAT3																								1						
w51 IL2RAT																														
w52 BID																														
w53 Apoptosis																														
w54 IL2RA																														
w55 CD45																														
w56 Stimuli2																														
w57 P2																														
w58 RANTES																														
w59 Proliferation																														
w60 P27																														

	a31 IFNG	a32 FLIP	a33 TNF	a34 MCL1	a35 NFKB	a36 A20	a37 TRADD	a38 IAP	a39 Caspase	a40 BclxL	a41 GZMB	a42 IL2	a43 LCK	a44 Stimuli	a45 TCR	a46 CTLA4	a47 JAK	a48 SOCS	a49 IFNGT	a50 STAT3	a51 IL2RAT	a52 BID	a53 Apoptosis	a54 IL2RA	a55 CD45	a56 Stimuli2	a57 P2	a58 RANTES	a59 Proliferation	a60 P27
w1 Cytoskeleton signalling																														
w2 FYN																														
w3 ZAP70													-1																	
w4 GRB2																														
w5 PLCG1																														
w6 PDGF																														
w7 PDGFR																														
w8 SPHK1																														
w9 S1P																														
w10 sFas																														
w11 Ceramide																														
w12 GPCR																														
w13 SMAD	-1																													
w14 RAS																														
w15 GAP																														
w16 MEK																														
w17 ERK																														
w18 P13K				1	1																									
w19 Fas																														
w20 Fast																														
w21 TAX																														
w22 TPL2					1																									
w23 DISC		-1		-1					1	-1																				
w24 FasL																														
w25 NFAT												1							1											
w26 CREB		1									1																			
w27 IL2RBT																														
w28 TBET											1	-1							1											
w29 IL2RB				1									1				1													
w30 IL15	1																	-1												
w31 IFNG		1									1						1										1			
w32 FLIP					1																									
w33 TNF							1																							
w34 MCL1																						-1								
w35 NFKB		1	1	1		1		1		1		1									1							1		
w36 A20							-1																							
w37 TRADD					1				1																					
w38 IAP					1		-1		-1																					
w39 Caspase																						1	1							
w40 BclxL																						-1								
w41 GZMB									1	-1												1								
w42 IL2	1																	-1			1			1						
w43 LCK																														
w44 Stimuli	1																													
w45 TCR													1		1	1														
w46 CTLA4															-1															
w47 JAK																		1		1										
w48 SOCS																	-1													
w49 IFNGT	1																													
w50 STAT3				1						1		1							1		1								1	1
w51 IL2RAT																								1						
w52 BID								-1	1	-1																				
w53 Apoptosis																							1							
w54 IL2RA																	1							1						
w55 CD45													1				-1													
w56 Stimuli2																											-1			
w57 P2	-1																										1			
w58 RANTES																	1													
w59 Proliferation																														
w60 P27																													-1	

Figure 116: The regulatory matrix for Figure 115.

9.1 Subsystems

9.1.1 Three-dimensional systems

Example 1. Consider $\mu_1 = 5$, $\mu_2 = 15$, $\mu_3 = 5$, $v_1 = v_2 = v_3 = 1$ and $\theta_1 = 1.2$, $\theta_2 = 0.5$, $\theta_3 = -0.6$. The regulatory matrix of the system (19) is

$$W = \begin{pmatrix} w_2a_3 & w_2a_4 & w_2a_5 \\ w_3a_3 & w_3a_4 & w_3a_5 \\ w_4a_3 & w_4a_4 & w_4a_5 \end{pmatrix}, \tag{71}$$

where $w_2a_3 = -1, w_2a_4 = w_2a_5 = 0, w_3a_3 = w_3a_5 = 0, w_3a_4 = 1, w_4a_3 = w_4a_4 = 0, w_4a_5 = 1$. The nullclines are depicted in Figure 117. There are exactly three critical points.

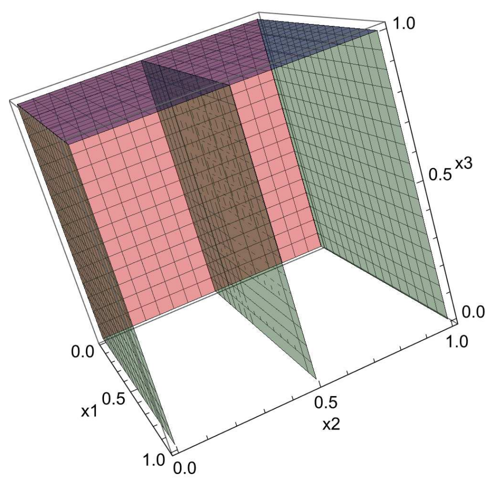

Figure 117: Visualization of nullclines (x_1 - red, x_2 - green, x_3 - blue) of the system(19) with the regulatory matrix (71).

The characteristic equation for critical point $(0.0024; 0.0006; 0.9997)$ is

$$-\lambda^3 + A\lambda^2 + B\lambda + C = 0, \tag{72}$$

where $A = -3.00215$, $B = -3.00419$ and $C = -1.00204$.

Solving the equation we have $\lambda_1 = -1.01218$, $\lambda_2 = -0.998321$ and $\lambda_3 = -0.991643$. The type of the critical point is a stable node.

The characteristic equation for critical point $(0.0024; 0.5; 0.9997)$ is (72), where $A = 0.739495$, $B = 4.5184$ and $C = 2.77883$.

Solving the equation we have $\lambda_1 = -1.01218$, $\lambda_2 = -0.998321$ and $\lambda_3 = 2.75$. The type of the critical point is a saddle.

The characteristic equation for critical point $(0.0024; 0.9994; 0.9997)$ is (72), where $A = -3.00215$, $B = -3.00419$ and $C = -1.00204$.

Solving the equation we have $\lambda_1 = -1.01218$, $\lambda_2 = -0.998321$ and $\lambda_3 = -0.991643$. The type of the critical point is a stable node.

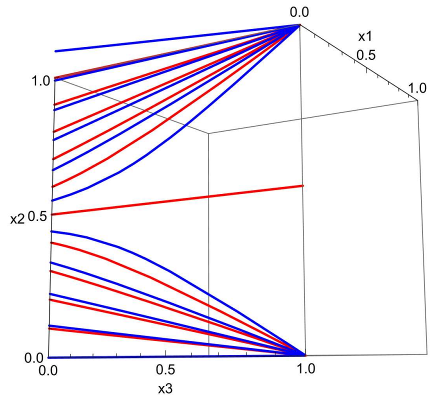

Figure 118: Visualization of two stable nodes and the saddle of the system(19) with the regulatory matrix (71).

Example 2. Consider $\mu_1 = 5$, $\mu_2 = 15$, $\mu_3 = 5$, $v_1 = v_2 = v_3 = 1$ and $\theta_1 = 1.2$, $\theta_2 = 0.5$, $\theta_3 = -0.6$. The regulatory matrix of the system (19) is

$$W = \begin{pmatrix} w_6a_7 & w_6a_8 & w_6a_9 \\ w_7a_7 & w_7a_8 & w_7a_9 \\ w_8a_7 & w_8a_8 & w_8a_9 \end{pmatrix}, \tag{73}$$

where $w_6a_7 = w_7a_8 = w_8a_9 = 1, w_6a_8 = w_6a_9 = w_7a_7 = w_7a_9 = w_8a_7 = w_8a_8 = 0$. The nullclines are depicted in Figure 119. There are exactly three critical points.

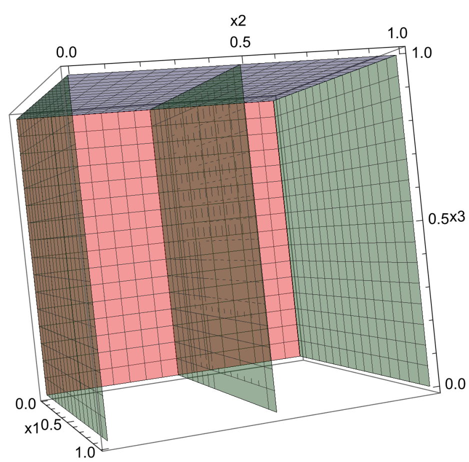

Figure 119: Visualization of nullclines (x_1 - red, x_2 - green, x_3 - blue) of the system(19) with the regulatory matrix (73).

The characteristic equation for critical point $(0.0025; 0.00056; 0.9997)$ is (72), where $A = -2.97748$, $B = -2.95509$ and $C = -0.977616$.

Solving the equation we have $\lambda_1 = -0.998321$, $\lambda_2 = -0.991643$ and $\lambda_3 = -0.987513$. The type of the critical point is a stable node.

The characteristic equation for critical point $(0.0025; 0.5; 0.9997)$ is (72), where $A = 0.764166$, $B = 4.47519$ and $C = 2.7111$.

Solving the equation we have $\lambda_1 = -0.998321$, $\lambda_2 = -0.987513$ and $\lambda_3 = 2.75$. The type of the critical point is a saddle.

The characteristic equation for critical point $(0.0025; 0.9994; 0.9997)$ is (72), where $A = -2.97748$, $B = -2.95509$ and $C = -0.977616$.

Solving the equation we have $\lambda_1 = -0.998321$, $\lambda_2 = -0.991643$ and $\lambda_3 = -0.987513$. The type of the critical point is a stable node.

Example 3. Consider $\mu_1 = 5$, $\mu_2 = 15$, $\mu_3 = 5$, $v_1 = v_2 = v_3 = 1$ and $\theta_1 = 1.2$, $\theta_2 = 0.5$, $\theta_3 = -0.6$. The regulatory matrix of the system (19) is

$$W = \begin{pmatrix} w_{14}a_{15} & w_{14}a_{16} & w_{14}a_{17} \\ w_{15}a_{15} & w_{15}a_{16} & w_{15}a_{17} \\ w_{16}a_{15} & w_{16}a_{16} & w_{16}a_{17} \end{pmatrix}, \tag{74}$$

where $w_{14}a_{15} = w_{14}a_{16} = 1, w_{14}a_{17} = 0, w_{15}a_{15} = 1, w_{15}a_{16} = w_{15}a_{17} = w_{16}a_{15} = w_{16}a_{16} = 0, w_{16}a_{17} = 1$. The nullclines are depicted in Figure 120. There are exactly three critical points.

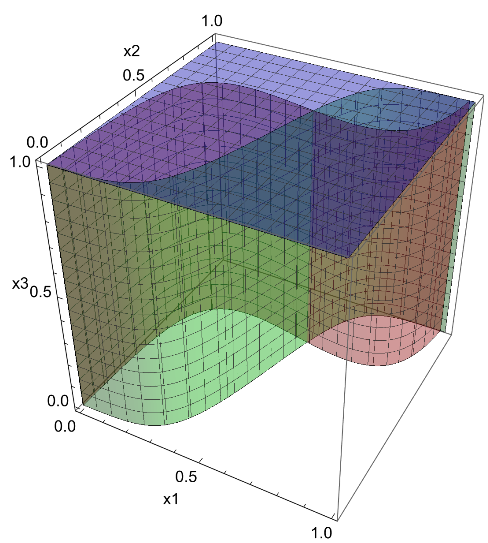

Figure 120: Visualization of nullclines (x_1 - red, x_2 - green, x_3 - blue) of the system(19) with the regulatory matrix (74).

The characteristic equation for critical point $(0.0025; 0.0006; 0.9997)$ is (72), where $A = -2.9858$, $B = -2.97151$ and $C = -0.985711$.

Solving the equation we have $\lambda_1 = -1.00586$, $\lambda_2 = -0.998321$ and $\lambda_3 = -0.981615$. The type of the critical point is a stable node.

The characteristic equation for critical point $(0.5532; 0.6895; 0.9997)$ is (72), where $A = -1.76247$, $B = 3.44158$ and $C = 4.19738$.

Solving the equation we have $\lambda_1 = -2.46784$, $\lambda_2 = -0.998321$ and $\lambda_3 = 1.70369$. The type of the critical point is a saddle.

The characteristic equation for critical point $(0.9801; 0.9993; 0.9997)$ is (72), where $A = -2.3731$, $B = 0.260488$ and $C = 1.63021$.

Solving the equation we have $\lambda_1 = -2.13841$, $\lambda_2 = -0.998321$ and $\lambda_3 = 0.763632$. The type of the critical point is a saddle.

Example 4. Consider $\mu_1 = 5$, $\mu_2 = 15$, $\mu_3 = 5$, $v_1 = v_2 = v_3 = 1$ and $\theta_1 = 1.2$, $\theta_2 = 0.5$, $\theta_3 = -0.6$. The regulatory matrix of the system (19) is

$$W = \begin{pmatrix} w_{17}a_{22} & w_{17}a_{23} & w_{17}a_{24} \\ w_{18}a_{22} & w_{18}a_{23} & w_{18}a_{24} \\ w_{19}a_{22} & w_{19}a_{23} & w_{19}a_{24} \end{pmatrix}, \tag{75}$$

where $w_{17}a_{22} = w_{17}a_{23} = 0, w_{17}a_{24} = w_{18}a_{22} = 1, w_{18}a_{23} = w_{18}a_{24} = w_{19}a_{22} = w_{19}a_{24} = 0, w_{19}a_{23} = 1$. The nullclines are depicted in Figure 121. There is one critical point.

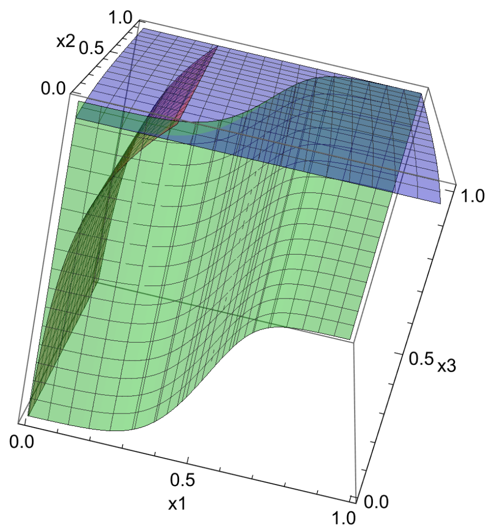

Figure 121: Visualization of nullclines (x_1 - red, x_2 - green, x_3 - blue) of the system(19) with the regulatory matrix (75).

The characteristic equation for critical point $(0.2281; 0.0167; 0.9562)$ is (72), where $A = -3$, $B = -3$ and $C = -0.954711$.

Solving the equation we have $\lambda_{1,2} = -1.17822 \pm 0.308693i$ and $\lambda_3 = -0.643552$. The type of critical point is a stable focus-node.

Example 5. Consider $\mu_1 = 5$, $\mu_2 = 15$, $\mu_3 = 5$, $v_1 = v_2 = v_3 = 1$ and $\theta_1 = 1.2$, $\theta_2 = 0.5$, $\theta_3 = -0.6$. The regulatory matrix of the system (19) is

$$W = \begin{pmatrix} w_{37}a_{35} & w_{37}a_{36} & w_{37}a_{37} \\ w_{38}a_{35} & w_{38}a_{36} & w_{38}a_{37} \\ w_{39}a_{35} & w_{39}a_{36} & w_{39}a_{37} \end{pmatrix}, \tag{76}$$

where $w_{37}a_{35} = w_{37}a_{37} = 0, w_{37}a_{36} = 1, w_{38}a_{35} = w_{38}a_{36} = 0, w_{38}a_{37} = -1, w_{39}a_{35} = 1, w_{39}a_{36} = w_{39}a_{37} = 0$. The nullclines are depicted in Figure 122. There is one critical point.

The characteristic equation for critical point $(0.0025; 0; 0.9531)$ is (72), where $A = -3$, $B = -3$ and $C = -1$.

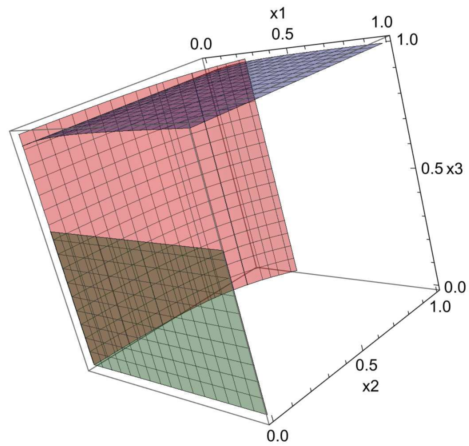

Figure 122: Visualization of nullclines (x_1 - red, x_2 - green, x_3 - blue) of the system(19) with the regulatory matrix (76).

Solving the equation we have $\lambda_{1,2} = -0.999879 \pm 0.000209324i$ and $\lambda_3 = -1.00024$. The type of the critical point is a stable focus-node.

Example 6. Consider $\mu_1 = 5$, $\mu_2 = 15$, $\mu_3 = 5$, $v_1 = v_2 = v_3 = 1$ and $\theta_1 = 1.2$, $\theta_2 = 0.5$, $\theta_3 = -0.6$. The regulatory matrix of the system (19) is

$$W = \begin{pmatrix} w_{37}a_{37} & w_{37}a_{38} & w_{37}a_{39} \\ w_{38}a_{37} & w_{38}a_{38} & w_{38}a_{39} \\ w_{39}a_{37} & w_{39}a_{38} & w_{39}a_{39} \end{pmatrix}, \tag{77}$$

where $w_{37}a_{37} = w_{37}a_{39} = 0, w_{37}a_{38} = 1, w_{38}a_{37} = -1, w_{38}a_{38} = w_{38}a_{39} = w_{39}a_{37} = w_{39}a_{38} = 0, w_{39}a_{39} = 1$. The nullclines are depicted in Figure 123. There is one critical point.

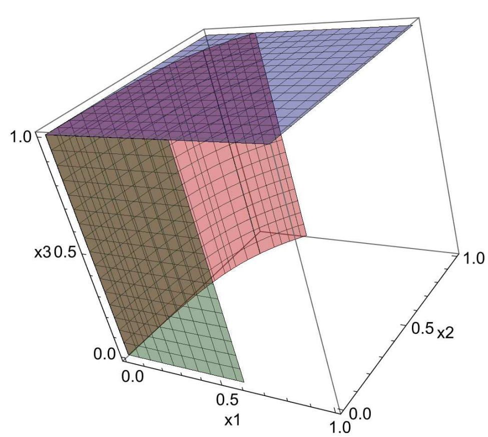

Figure 123: Visualization of nullclines (x_1 - red, x_2 - green, x_3 - blue) of the system(19) with the regulatory matrix (77).

The characteristic equation for critical point $(0.0025; 0.0005; 0.9997)$ is (72), where $A = -2.99832$, $B = -2.99674$ and $C = -0.99842$.

Solving the equation we have $\lambda_{1,2} = -1 \pm 0.00993657i$ and $\lambda_3 = -0.998321$. The type of the critical point is a stable focus-node.

9.1.2 Four-dimensional systems

Example 1. The regulatory matrix is

$$W = \begin{pmatrix} w_6a_7 & w_6a_8 & w_6a_9 & w_6a_{10} \\ w_7a_7 & w_7a_8 & w_7a_9 & w_7a_{10} \\ w_8a_7 & w_8a_8 & w_8a_9 & w_8a_{10} \\ w_9a_7 & w_9a_8 & w_9a_9 & w_9a_{10} \end{pmatrix},$$

where $w_6a_7 = w_7a_8 = w_8a_9 = w_9a_7 = w_9a_{10} = 1, w_6a_8 = w_6a_9 = w_6a_{10} = w_7a_7 = w_7a_9 = w_7a_{10} = w_8a_7 = w_8a_8 = w_8a_{10} = w_9a_8 = w_9a_9 = 0$ and $v_1 = v_2 = v_3 = v_4 = 1$, $\mu_1 = 5, \mu_2 = 15, \mu_3 = 5, \mu_4 = 5$, $\theta_1 = 1.2$, $\theta_2 = 0.5$, $\theta_3 = -0.6$, $\theta_4 = -0.2$.

The first critical point is $(0.0025; 0.00056; 0.9997; 0.9975)$. The standard linearization analysis provides the characteristic numbers $\lambda_1 = -0.998321$, $\lambda_2 = -0.991643$, $\lambda_3 = -0.987669$ and $\lambda_4 = -0.987513$. The type of the critical point is a 4D stable node.

The second critical point is $(0.00250369; 0.5; 0.999664; 0.997528)$. The standard linearization analysis provides the characteristic numbers $\lambda_1 = -0.998321$, $\lambda_2 = -0.987669$, $\lambda_3 = -0.987513$ and $\lambda_4 = 2.75$. The type of the critical point is a saddle.

The third critical point is $(0.00250369; 0.999443; 0.999664; 0.997528)$. The standard linearization analysis provides the characteristic numbers $\lambda_1 = -0.998321$, $\lambda_2 = -0.991643$, $\lambda_3 = -0.987669$ and $\lambda_4 = -0.987513$. The type of the critical point is a 4D stable node.

Example 2. The regulatory matrix is

$$W = \begin{pmatrix} w_{37}a_{37} & w_{37}a_{38} & w_{37}a_{39} & w_{37}a_{40} \\ w_{38}a_{37} & w_{38}a_{38} & w_{38}a_{39} & w_{38}a_{40} \\ w_{39}a_{37} & w_{39}a_{38} & w_{39}a_{39} & w_{39}a_{40} \\ w_{40}a_{37} & w_{40}a_{38} & w_{40}a_{39} & w_{40}a_{40} \end{pmatrix},$$

where $w_{37}a_{37} = w_{37}a_{39} = w_{38}a_{38} = w_{38}a_{39} = w_{38}a_{40} = w_{39}a_{37} = w_{39}a_{38} = w_{39}a_{40} = w_{39}a_{40} = w_{40}a_{40} = 0, w_{37}a_{38} = w_{37}a_{40} = w_{39}a_{39} = 1 w_{38}a_{37} = w_{40}a_{37} = w_{40}a_{39} = -1$ and $v_1 = v_2 = v_3 = v_4 = 1$, $\mu_1 = 5, \mu_2 = 15, \mu_3 = 5, \mu_4 = 5$, $\theta_1 = 1.2$, $\theta_2 = 0.5$, $\theta_3 = -0.6$, $\theta_4 = -0.2$.

The critical point is $(0.0027; 0.0005; 0.9997; 0.01778)$. The standard linearization analysis provides the characteristic numbers $\lambda_1 = -1$, $\lambda_{2,3} = -1 \pm 0.03587i$ and $\lambda_4 = -0.998321$. The type of the critical point is a stable focus-node.

Example 3. The regulatory matrix is

$$W = \begin{pmatrix} w_{37}a_{35} & w_{37}a_{36} & w_{37}a_{37} & w_{37}a_{38} \\ w_{38}a_{35} & w_{38}a_{36} & w_{38}a_{37} & w_{38}a_{38} \\ w_{39}a_{35} & w_{39}a_{36} & w_{39}a_{37} & w_{39}a_{38} \\ w_{40}a_{35} & w_{40}a_{36} & w_{40}a_{37} & w_{40}a_{38} \end{pmatrix},$$

where $w_{37}a_{35} = w_{37}a_{37} = w_{38}a_{35} = w_{38}a_{36} = w_{38}a_{38} = w_{39}a_{36} = w_{39}a_{37} = w_{39}a_{38} = w_{40}a_{36} = w_{40}a_{38} = 0, w_{37}a_{36} = w_{37}a_{38} = w_{39}a_{35} = w_{40}a_{35} = 1, w_{38}a_{37} = w_{40}a_{37} = -1$ and $v_1 = v_2 = v_3 = v_4 = 1$, $\mu_1 = 5, \mu_2 = 15, \mu_3 = 5, \mu_4 = 5$, $\theta_1 = 1.2$, $\theta_2 = 0.5$, $\theta_3 = -0.6$, $\theta_4 = -0.2$.

The critical point is $(0.0028; 0; 0.9532; 0.0229)$. The standard linearization analysis provides the characteristic numbers $\lambda_1 = -1.07749$, $\lambda_{2,3} = -0.9613 \pm 0.05437i$ and $\lambda_4 = -1$. The type of the critical point is a stable focus-node.

Example 4. The regulatory matrix is

$$W = \begin{pmatrix} w_{41}a_{51} & w_{41}a_{52} & w_{41}a_{53} & w_{41}a_{54} \\ w_{42}a_{51} & w_{42}a_{52} & w_{42}a_{53} & w_{42}a_{54} \\ w_{43}a_{51} & w_{43}a_{52} & w_{43}a_{53} & w_{43}a_{54} \\ w_{44}a_{51} & w_{44}a_{52} & w_{44}a_{53} & w_{44}a_{54} \end{pmatrix},$$

where $w_{41}a_{51} = w_{41}a_{54} = w_{42}a_{51} = w_{42}a_{53} = w_{42}a_{54} = w_{43}a_{51} = w_{43}a_{53} = w_{43}a_{54} = w_{44}a_{52} = w_{44}a_{53} = 0, w_{41}a_{52} = w_{43}a_{52} = w_{44}a_{51} = w_{44}a_{54} = 1, w_{41}a_{53} = w_{42}a_{52} = -1$ and $v_1 = v_2 = v_3 = v_4 = 1,\ \mu_1 = 5, \mu_2 = 15, \mu_3 = 5, \mu_4 = 5,\ \theta_1 = 1.2,\ \theta_2 = 0.5,\ \theta_3 = -0.6,\ \theta_4 = -0.2$.

The critical point is $(0.000021; 0.00055; 0.9527; 0.9975)$. The standard linearization analysis provides the characteristic numbers $\lambda_1 = -1.00822$, $\lambda_2 = -1$, $\lambda_3 = -1$ and $\lambda_4 = -0.987514$. The type of the critical point is a 4D stable node.

10 Conclusions

Main results of the Doctoral thesis are:

- Systems of orders two and three are considered with the regulatory matrices of different structures. The number and the character of critical points are considered.
- For three-dimensional systems and four-dimensional systems chaotic attractors were considered. Examples were constructed. In the thesis for Lyapunov exponents calculation the package "lce.m for Mathematica" was used. Another Wolfram Mathematica program "Lynch-DSAM.nb" was also used to check the correctness of Lyapunov exponents calculation.
- Formulas for characteristic numbers of critical points for four-dimensional systems were obtained. Examples of 4D systems with stable equilibria were constructed.
- Neuronal networks were considered and similarity with the corresponding ODE-type models was detected.
- Examples of 5D were constructed. These systems possess periodic attractors. The visualization of attractors of 5D by projecting them into lower dimension subspaces and considering graphs of components of solutions was made.
- Examples of 6D systems were constructed. These systems possess periodic attractors and exhibit irregular behavior of solutions. The visualization of attractors of 6D systems by projecting them into lower dimension subspaces and considering graphs of components of solutions was made.
- Sixty-dimensional system was considered. The graph of 60×60 matrix with the program Graphia was constructed. Some subsystems of the 60D system were considered.

The study of gene regulatory networks is important for human life and activity. Both for the treatment of various diseases such as leukemia, multiple sclerosis, and Alzheimer's, and for describing problems and their solutions in economics, psychology, politics, and many other areas. The more there is in a system of equations, the more similar it is to the gene network that occurs in life. The main task is to continue the research and find methods for studying systems with a large number of equations.

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